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SINGULARITIES OF AN INCLINED CRACK TERMINATING AT AN ANISOTROPIC BIMATERIAL INTERFACE

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Abstract—Characteristic equation for the stress singularities of the form $r^{\lambda-1}$, $0 < \mathrm{Re}[\lambda] < 1$, for an inclined crack terminating at an interface between two anisotropic media is derived. Explicit dependence of λ on the material parameters is further carried out for materials, both having properties of orthotropy. Characteristic equation corresponding to any degenerate bimaterial problem is obtainable from present results. Numerical results of λ are presented for certain kinds of bimaterial problems, and the influences of the material parameters, the meeting angles and the material alignments on λ are examined. The role of the parameter β_0 playing in the oscillatory behavior is also discussed. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Analysis of stress singularities of an inclined crack terminating at an interface between two isotropic materials has been investigated by Bogy (1971). Very detailed investigations of the dependence of the stress singularities on the Dundurs' constants α and β were given. The special case of crack terminating normally at the interface has been treated very early by Zak and Williams (1963) and later on by Cook and Erdogan (1972), Erdogan and Biricikoglu (1973). All these studies are focused on isotropic bimaterials. Singularities of cracks normally terminating at an interface between two aligned orthotropic materials have been studied by Delale and Erdogan (1979), and Gupta *et al.* (1992). Only the deformation for symmetric mode is considered by them. Hence, only one real root is found for the power of singularities λ for most of the material combinations chosen in their studies. Ting and Hoang (1984) investigated their problems (Delale and Erdogan (1979), and Gupta *et al.* (1992)) but with materials both having the properties of general anisotropy. Orientation of the crack relative to the interface remains normal in the analysis. Recently, Sung and Liou (1996) have also investigated the problem similar to that considered by Ting and Hoang (1984). A more convenient characteristic equation has been set up from the consideration of the singular behaviors of a system of singular integral equations. In terms of Krenk's parameters some features of the characteristic roots related to orthotropic bimaterials are further explored (Sung and Liou, 1996).

In the present analysis, characteristic equation for singularities λ, $0 < \mathrm{Re}[\lambda] < 1$, of an inclined crack terminating at an anisotropic interface is derived. The method used by Wu and Chang (1993) in the analysis of a wedge problem interacting with singularities is essentially followed. Characteristic equation corresponding to the special case treated by Sung and Liou (1996) is recovered from the present result by letter $\varphi = \pi/2$, the meeting angle of the inclined crack relative to the interface. Following that paper (Sung and Liou, 1996), a special kind of alignments of orthotropic materials on both sides of the interface is then considered. The explicit form of the characteristic equation expressed in terms of material parameters is obtained. Characteristic equations corresponding to any degenerate bimaterial problem are obtainable from present results through the appropriate limiting process. When crack meets the interface at $\varphi = 0$ or π, i.e., an interfacial crack problem, the oscillatory index ε is found to depend on four generalized Dundurs' constants, β_1, β_2,

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λ_1, and λ_2, only, and is independent of the rest of the generalized Dundurs' constants, α_1 and α_2. The oscillatory index will vanish if $\beta_1 = \beta_2 = 0$ (or $\beta_0 = 0$). Numerical results of the roots λ are first presented for the problem of an isotropic material joined to an orthotropic material for different meeting angles. The parameters α_0 and β_0, which reduced, respectively, to Dundurs' constants α and β for two isotropic materials, are focused on in the studies of singular natures and the role of α_0 in the measure of dissimilarity of two materials is discussed. Next the problem of a mismatch problem, composed by two but the same orthotropic materials, is analyzed. The effect of mismatch angle on the roots λ is investigated. Material parameter κ is found to have little influence on the magnitudes of the singularity for mismatch problem. Finally, a real graphite–epoxy composite where a crack is embedded in either one of the materials is analyzed. From numerical investigations, the role parameter β_0 possibly played in the oscillatory behavior for bimaterials is discussed.

2. STROH FORMALISM

A two-dimensional deformation of a linear elastic solid whose field quantities are only functions of x_1 and x_2 is considered. The general expressions for the displacement **u** and stress function ϕ for such a deformation are (Eshelby *et al.* (1953), Stroh (1958))

$$\mathbf{u} = 2\,\mathrm{Re}\,\{\mathbf{Af(z)}\} \tag{1}$$

$$\phi = 2\,\mathrm{Re}\,\{\mathbf{Bf(z)}\} \tag{2}$$

where Re { } denotes real part,

$$\mathbf{f(z)} = (f_1(z_1), f_2(z_2), f_3(z_3))^{\mathrm{T}} \tag{3}$$

with $z_k = x_1 + p_k x_2$, $(k = 1,2,3)$. Superscript T represents transpose. Matrix **A** with components denoted by a_{kj} and constants p_k are determined from the following eigenvalue problem

$$\{c_{i1k1} + p_j(c_{i1k2} + c_{i2k1}) + p_j^2 c_{i2k2}\}a_{kj} = 0 \quad (\text{no sum on } j) \tag{4}$$

where c_{ijkl} are the elastic constants. Without loss of generality, one may take the imaginary part of p_k to be positive. Matrix **B** in eqn (2) is defined by

$$\mathbf{B} = \mathbf{R}^{\mathrm{T}}\mathbf{A} + \mathbf{TAP} \tag{5}$$

where

$$R_{ik} = c_{i1k2} \tag{6}$$

$$T_{ik} = c_{i2k2} \tag{7}$$

and $\mathbf{P} = \mathrm{diag}\langle p_1, p_2, p_3\rangle$. (For a more detailed description of the function **f(z)** and the physical meaning of matrices **A** and **B**, please refer to the paper by, e.g., Ting (1986.)) Stress function ϕ is related to the stress components by

$$\mathbf{t}_1 = (\sigma_{11}, \sigma_{12}, \sigma_{13})^{\mathrm{T}} = \frac{-\partial\phi}{\partial x_2} = -2\,\mathrm{Re}\,\{\mathbf{BPf}'(\mathbf{z})\} \tag{8}$$

$$\mathbf{t}_2 = (\sigma_{21}, \sigma_{22}, \sigma_{23})^{\mathrm{T}} = \frac{\partial\phi}{\partial x_1} = 2\,\mathrm{Re}\,\{\mathbf{Bf}'(\mathbf{z})\} \tag{9}$$

where

$$\mathbf{f}'(\mathbf{z}) = \frac{\mathrm{d}\mathbf{f}(\mathbf{z})}{\mathrm{d}\mathbf{z}} = \left(\frac{\mathrm{d}f_1(z_1)}{\mathrm{d}z_1}, \frac{\mathrm{d}f_2(z_2)}{\mathrm{d}z_2}, \frac{\mathrm{d}f_3(z_3)}{\mathrm{d}z_3}\right)^{\mathrm{T}}.$$

A more general expression (see, e.g. Ting, 1986)

$$\mathbf{t}_{\mathbf{n}} = \frac{\partial \phi}{\partial s} \tag{10}$$

can be established where s is the arc length and $\mathbf{n}$ the unit outward normal vector. Matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the following orthogonality relations (Stroh (1958), Chadwick and Smith (1977)):

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \bar{\mathbf{B}} \\ \mathbf{A} & \bar{\mathbf{A}} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \tag{11}$$

where $\mathbf{I}$ is a 3×3 unit matrix and a bar over a quantity represents the conjugate of that quantity.

3. CHARACTERISTIC EQUATION FOR ANISOTROPIC BIMATERIALS

In this section, we will develop the characteristic equation for an inclined crack terminating at the interface between two anisotropic media. We follow the approach developed by Wu and Chang (1993) to select the vector functions $\mathbf{f(z)}$ so that the traction-free conditions on the crack faces are automatically satisfied. Hence, the remaining conditions to be enforced for setting up the characteristic equation are the continuity conditions of the displacements and stresses on the interface which will make the derivation of the characteristic equation easier. The functions $\mathbf{f(z)}$ defined in eqn (1) in three regions (as shown in Fig. 1) are selected as

$$\mathbf{f}^{[1]} = \frac{c_1^{[1]}}{2}\left(\frac{r}{\hat{r}}\right)^{\lambda} \Lambda^{(1)}(\theta, 2\pi-\varphi, \lambda)\mathbf{q}_1^{[1]} + \frac{\bar{c}_1^{[1]}}{2}\left(\frac{r}{\hat{r}}\right)^{\bar{\lambda}} \Lambda^{(1)}(\theta, 2\pi-\varphi, \bar{\lambda})\bar{\mathbf{q}}_1^{[1]}$$
$$+ \frac{c_2^{[1]}}{2}\left(\frac{r}{\hat{r}}\right)^{\lambda} \Lambda^{(1)}(\theta, -\varphi, \lambda)\mathbf{q}_2^{[1]} + \frac{\bar{c}_2^{[1]}}{2}\left(\frac{r}{\hat{r}}\right)^{\bar{\lambda}} \Lambda^{(1)}(\theta, -\varphi, \bar{\lambda})\bar{\mathbf{q}}_2^{[1]} \tag{12.a}$$

$$\mathbf{f}^{[2]} = \frac{c^{[2]}}{2}\left(\frac{r}{\hat{r}}\right)^{\lambda} \Lambda^{(2)}(\theta, 2\pi-\varphi, \lambda)\mathbf{q}^{[2]} + \frac{\bar{c}^{[2]}}{2}\left(\frac{r}{\hat{r}}\right)^{\bar{\lambda}} \Lambda^{(2)}(\theta, 2\pi-\varphi, \bar{\lambda})\bar{\mathbf{q}}^{[2]} \tag{12.b}$$

$$\mathbf{f}^{[3]} = \frac{c^{[3]}}{2}\left(\frac{r}{\hat{r}}\right)^{\lambda} \Lambda^{(2)}(\theta, -\varphi, \lambda)\mathbf{q}^{[3]} + \frac{\bar{c}^{[3]}}{2}\left(\frac{r}{\hat{r}}\right)^{\bar{\lambda}} \Lambda^{(2)}(\theta, -\varphi, \bar{\lambda})\bar{\mathbf{q}}^{[3]} \tag{12.c}$$

where $c_1^{[1]}$, $c_2^{[1]}$, $c^{[2]}$ and $c^{[3]}$ are unknown complex constants, $\mathbf{q}^{[1]}$, $\mathbf{q}^{[2]}$ and $\mathbf{q}^{[3]}$ are complex vectors to be determined, $\hat{r}$ is a characteristic length scale and $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are given as follows:

$$\Lambda^{(1)}(\theta, \theta^*, \lambda) = (\mathbf{B}^{(1)^{\mathrm{T}}}\mathbf{A}^{(2)} + \mathbf{A}^{(1)^{\mathrm{T}}}\mathbf{B}^{(2)}) \left\langle \left(\frac{\zeta^{(1)}(\theta)}{-\zeta^{(2)}(\theta^*)}\right)^{\lambda} \right\rangle \mathbf{B}^{(2)^{-1}}$$

$$\left\langle \left(\frac{\zeta^{(1)}(\theta)}{-\zeta^{(2)}(\theta^*)}\right)^{\lambda} \right\rangle = \mathrm{diag}\left[\left(\frac{\zeta_1^{(1)}(\theta)}{-\zeta_1^{(2)}(\theta^*)}\right)^{\lambda}, \left(\frac{\zeta_2^{(1)}(\theta)}{-\zeta_2^{(2)}(\theta^*)}\right)^{\lambda}, \left(\frac{\zeta_3^{(1)}(\theta)}{-\zeta_3^{(2)}(\theta^*)}\right)^{\lambda}\right] \tag{13.a}$$

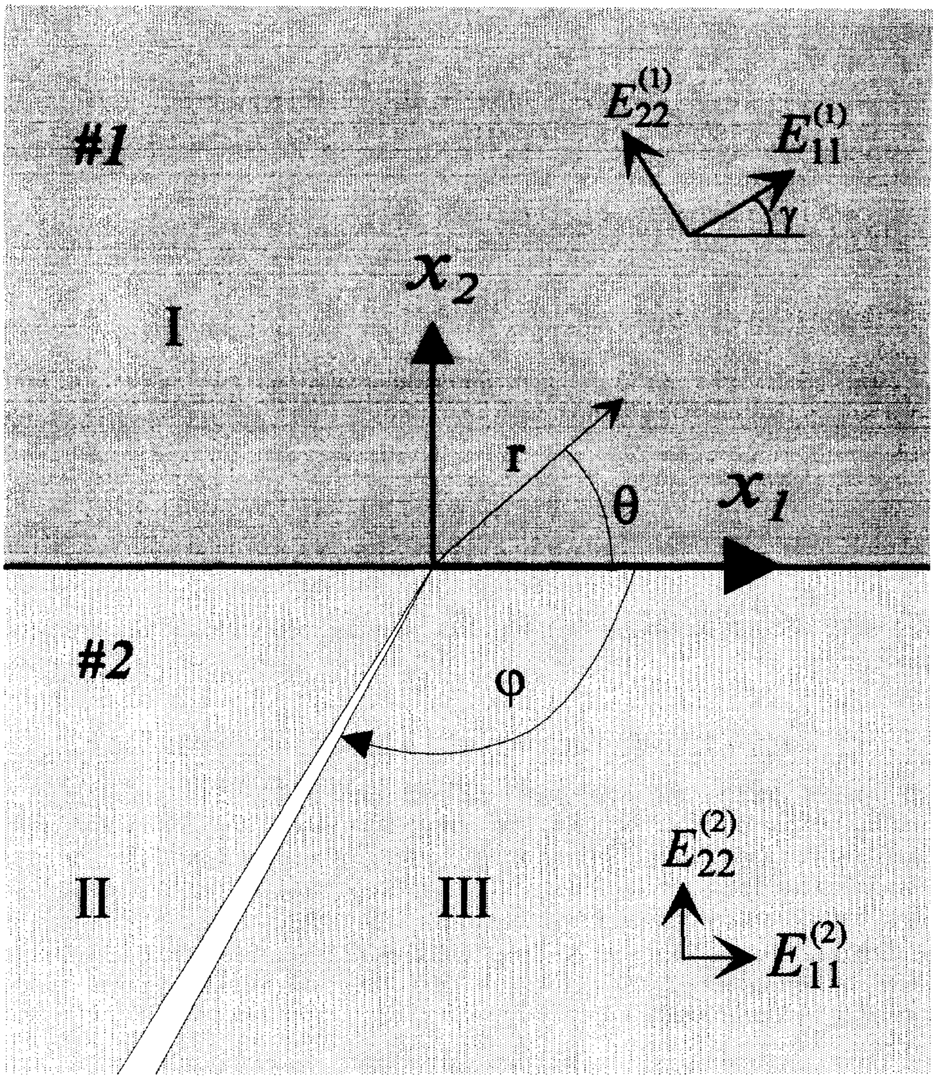


Fig. 1. An inclined crack terminating at an anisotropic interface.

$$\Lambda^{(2)}(\theta,\theta^*,\lambda) = \left\langle \left(\frac{\zeta^{(2)}(\theta)}{\zeta^{(2)}(\theta^*)}\right)^{\lambda} \right\rangle \mathbf{B}^{(2)^{\mathrm{T}}}$$

$$= \mathrm{diag}\left[\left(\frac{\zeta_1^{(2)}(\theta)}{\zeta_1^{(2)}(\theta^*)}\right)^{\lambda}, \left(\frac{\zeta_2^{(2)}(\theta)}{\zeta_2^{(2)}(\theta^*)}\right)^{\lambda}, \left(\frac{\zeta_3^{(2)}(\theta)}{\zeta_3^{(2)}(\theta^*)}\right)^{\lambda}\right]\mathbf{B}^{(2)^{\mathrm{T}}} \tag{13.b}$$

where

$$\zeta_j^{(\alpha)}(\theta) = \cos\theta + p_j^{(\alpha)}\sin\theta. \tag{13.c}$$

Bracketed superscript $[i]$ represents the region's number ($i = 1, 2, 3$) while a superscript with parentheses (α) denotes the material's number ($\alpha = 1, 2$). Substituting eqns (12.a, b, c) and (13.a, b) into eqns (1), (2) and (10), the corresponding displacements, stress functions and tractions in three regions are given by

$$\mathbf{u}^{[1]} = \mathrm{Re}\left\{\left(\frac{r}{\hat{r}}\right)^{\lambda}[c_1^{[1]}\mathbf{h}_1^{[1]}(\theta, 2\pi-\varphi, \lambda) + c_2^{[1]}\mathbf{h}_2^{[1]}(\theta, -\varphi, \lambda)]\right\} \tag{14.a}$$

$$\phi^{[1]} = \mathrm{Re}\left\{\left(\frac{r}{\hat{r}}\right)^{\lambda}[c_1^{[1]}\mathbf{g}_1^{[1]}(\theta, 2\pi-\varphi, \lambda) + c_2^{[1]}\mathbf{g}_2^{[1]}(\theta, -\varphi, \lambda)]\right\} \tag{14.b}$$

$$\mathbf{t}_r^{[1]} = -\frac{\partial \phi^{[1]}}{\partial r} = -\mathrm{Re}\left\{\lambda\left(\frac{r}{\hat{r}}\right)^{\lambda-1}[c_1^{[1]}\mathbf{g}_1^{[1]}(\theta, 2\pi-\varphi, \lambda) + c_2^{[1]}\mathbf{g}_2^{[1]}(\theta, -\varphi, \lambda)]\right\} \tag{14.c}$$

$$\mathbf{u}^{[2]} = \mathrm{Re}\left\{c^{[2]}\left(\frac{r}{\hat{r}}\right)^{\lambda}\mathbf{h}^{[2]}(\theta, 2\pi-\varphi, \lambda)\right\} \tag{15.a}$$

$$\phi^{[2]} = \mathrm{Re}\left\{c^{[2]}\left(\frac{r}{\hat{r}}\right)^{\lambda}\mathbf{g}^{[2]}(\theta, 2\pi-\varphi, \lambda)\right\} \tag{15.b}$$

$$\mathbf{t}_r^{[2]} = -\frac{\partial \phi^{[2]}}{\partial r} = -\mathrm{Re}\left\{c^{[2]}\lambda\left(\frac{r}{\hat{r}}\right)^{\lambda-1}\mathbf{g}^{[2]}(\theta, 2\pi-\varphi, \lambda)\right\} \tag{15.c}$$

$$\mathbf{u}^{[3]} = \mathrm{Re}\left\{c^{[3]}\left(\frac{r}{\hat{r}}\right)^{\lambda}\mathbf{h}^{[3]}(\theta, -\varphi, \lambda)\right\} \tag{16.a}$$

$$\phi^{[3]} = \mathrm{Re}\left\{c^{[3]}\left(\frac{r}{\hat{r}}\right)^{\lambda}\mathbf{g}^{[3]}(\theta, -\varphi, \lambda)\right\} \tag{16.b}$$

$$\mathbf{t}_r^{[3]} = -\frac{\partial \phi^{[3]}}{\partial r} = -\mathrm{Re}\left\{c^{[3]}\lambda\left(\frac{r}{\hat{r}}\right)^{\lambda-1}\mathbf{g}^{[3]}(\theta, -\varphi, \lambda)\right\} \tag{16.c}$$

where

$$\begin{aligned}
\mathbf{h}_1^{[1]}(\theta, 2\pi-\varphi, \lambda) &= (\mathbf{A}^{(1)}\Lambda^{(1)}(\theta, 2\pi-\varphi, \lambda) + \overline{\mathbf{A}^{(1)}\Lambda^{(1)}(\theta, 2\pi-\varphi, \bar{\lambda})})\mathbf{q}_1^{[1]} \\
\mathbf{g}_1^{[1]}(\theta, 2\pi-\varphi, \lambda) &= (\mathbf{B}^{(1)}\Lambda^{(1)}(\theta, 2\pi-\varphi, \lambda) + \overline{\mathbf{B}^{(1)}\Lambda^{(1)}(\theta, 2\pi-\varphi, \bar{\lambda})})\mathbf{q}_1^{[1]} \\
\mathbf{h}_2^{[1]}(\theta, -\varphi, \lambda) &= (\mathbf{A}^{(1)}\Lambda^{(1)}(\theta, -\varphi, \lambda) + \overline{\mathbf{A}^{(1)}\Lambda^{(1)}(\theta, -\varphi, \bar{\lambda})})\mathbf{q}_2^{[1]} \\
\mathbf{g}_2^{[1]}(\theta, -\varphi, \lambda) &= (\mathbf{B}^{(1)}\Lambda^{(1)}(\theta, -\varphi, \lambda) + \overline{\mathbf{B}^{(1)}\Lambda^{(1)}(\theta, -\varphi, \bar{\lambda})})\mathbf{q}_2^{[1]} \\
\mathbf{h}^{[2]}(\theta, 2\pi-\varphi, \lambda) &= (\mathbf{A}^{(2)}\Lambda^{(2)}(\theta, 2\pi-\varphi, \lambda) + \overline{\mathbf{A}^{(2)}\Lambda^{(2)}(\theta, 2\pi-\varphi, \bar{\lambda})})\mathbf{q}^{[2]} \\
\mathbf{g}^{[2]}(\theta, 2\pi-\varphi, \lambda) &= (\mathbf{B}^{(2)}\Lambda^{(2)}(\theta, 2\pi-\varphi, \lambda) + \overline{\mathbf{B}^{(2)}\Lambda^{(2)}(\theta, 2\pi-\varphi, \bar{\lambda})})\mathbf{q}^{[2]} \\
\mathbf{h}^{[3]}(\theta, -\varphi, \lambda) &= (\mathbf{A}^{(2)}\Lambda^{(2)}(\theta, -\varphi, \lambda) + \overline{\mathbf{A}^{(2)}\Lambda^{(2)}(\theta, -\varphi, \bar{\lambda})})\mathbf{q}^{[3]} \\
\mathbf{g}^{[3]}(\theta, -\varphi, \lambda) &= (\mathbf{B}^{(2)}\Lambda^{(2)}(\theta, -\varphi, \lambda) + \overline{\mathbf{B}^{(2)}\Lambda^{(2)}(\theta, -\varphi, \bar{\lambda})})\mathbf{q}^{[3]}.
\end{aligned} \tag{17}$$

It is seen from eqns (15.c) and (16.c) that the vanishing tractions on the planes $\theta = 2\pi-\varphi$ and $\theta = -\varphi$ are both satisfied. There remains the continuity conditions of tractions and displacements along the interfaces (i.e., $\theta = 0$ and $\theta = \pi$) that have to be satisfied. Therefore, enforcing these continuity conditions, one obtains the following expressions:

$$\begin{aligned}
&\mathrm{Re}\,\{[\mathbf{A}^{(1)}\Lambda^{(1)}(\pi, 2\pi-\varphi, \lambda) + \overline{\mathbf{A}^{(1)}\Lambda^{(1)}(\pi, 2\pi-\varphi, \bar{\lambda})}]\hat{\mathbf{q}}_1^{[1]} \\
&\quad + [\mathbf{A}^{(1)}\Lambda^{(1)}(\pi, -\varphi, \lambda) + \overline{\mathbf{A}^{(1)}\Lambda^{(1)}(\pi, -\varphi, \bar{\lambda})}]\hat{\mathbf{q}}_2^{[1]}\} \\
&= \mathrm{Re}\,\{[\mathbf{A}^{(2)}\Lambda^{(2)}(\pi, 2\pi-\varphi, \lambda) + \overline{\mathbf{A}^{(2)}\Lambda^{(2)}(\pi, 2\pi-\varphi, \bar{\lambda})}]\hat{\mathbf{q}}^{[2]}\}
\end{aligned} \tag{18.a}$$

$$\begin{aligned}
&\mathrm{Re}\,\{[\mathbf{B}^{(1)}\Lambda^{(1)}(\pi, 2\pi-\varphi, \lambda) + \overline{\mathbf{B}^{(1)}\Lambda^{(1)}(\pi, 2\pi-\varphi, \bar{\lambda})}]\hat{\mathbf{q}}_1^{[1]} \\
&\quad + [\mathbf{B}^{(1)}\Lambda^{(1)}(\pi, -\varphi, \lambda) + \overline{\mathbf{B}^{(1)}\Lambda^{(1)}(\pi, -\varphi, \bar{\lambda})}]\hat{\mathbf{q}}_2^{[1]}\} \\
&= \mathrm{Re}\,\{[\mathbf{B}^{(2)}\Lambda^{(2)}(\pi, 2\pi-\varphi, \lambda) + \overline{\mathbf{B}^{(2)}\Lambda^{(2)}(\pi, 2\pi-\varphi, \bar{\lambda})}]\hat{\mathbf{q}}^{[2]}\}
\end{aligned} \tag{18.b}$$

$$\mathrm{Re}\,\{[\mathbf{A}^{(1)}\Lambda^{(1)}(0, 2\pi-\varphi, \lambda) + \overline{\mathbf{A}^{(1)}\Lambda^{(1)}(0, 2\pi-\varphi, \bar{\lambda})}]\hat{\mathbf{q}}_1^{[1]}$$

$$+[\mathbf{A}^{(1)}\Lambda^{(1)}(0,-\varphi,\lambda)+\overline{\mathbf{A}^{(1)}\Lambda^{(1)}(0,-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}_2^{[1]}\}$$
$$= \mathrm{Re}\,\{[\mathbf{A}^{(2)}\Lambda^{(2)}(0,-\varphi,\lambda)+\overline{\mathbf{A}^{(2)}\Lambda^{(2)}(0,-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}^{[3]}\} \tag{18.c}$$
$$\mathrm{Re}\,\{[\mathbf{B}^{(1)}\Lambda^{(1)}(0,2\pi-\varphi,\lambda)+\overline{\mathbf{B}^{(1)}\Lambda^{(1)}(0,2\pi-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}_1^{[1]}$$
$$+[\mathbf{B}^{(1)}\Lambda^{(1)}(0,-\varphi,\lambda)+\overline{\mathbf{B}^{(1)}\Lambda^{(1)}(0,-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}_2^{[1]}\}$$
$$= \mathrm{Re}\,\{[\mathbf{B}^{(2)}\Lambda^{(2)}(0,-\varphi,\lambda)+\overline{\mathbf{B}^{(2)}\Lambda^{(2)}(0,-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}^{[3]}\} \tag{18.d}$$

where $\hat{\mathbf{q}}_1^{[1]} = c_1^{[1]}\mathbf{q}_1^{[1]}$, $\hat{\mathbf{q}}_2^{[1]} = c_2^{[1]}\mathbf{q}_2^{[1]}$, $\hat{\mathbf{q}}^{[2]} = c^{[2]}\mathbf{q}^{[2]}$ and $\hat{\mathbf{q}}^{[3]} = c^{[3]}\mathbf{q}^{[3]}$. Since the values of function $\Lambda^{(1)}$ (or $\Lambda^{(2)}$) at $\theta = 0$ and $\theta = \pi$ are equal, only two of the above equations need to be considered. For example, take eqns (18.a) and (18.b) for further consideration. Using results of eqn (11), these two equations can be rewritten as

$$\Lambda^{(1)}(\pi,2\pi-\varphi,\lambda)\hat{\mathbf{q}}^{[1]} = [(\mathbf{B}^{(1)^{\mathrm{T}}}\mathbf{A}^{(2)}+\mathbf{A}^{(1)^{\mathrm{T}}}\mathbf{B}^{(2)})\Lambda^{(2)}(\pi,2\pi-\varphi,\lambda)$$
$$+(\mathbf{B}^{(1)^{\mathrm{T}}}\overline{\mathbf{A}^{(2)}}+\mathbf{A}^{(1)^{\mathrm{T}}}\overline{\mathbf{B}^{(2)}})\overline{\Lambda^{(2)}(\pi,2\pi-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}^{[2]} \tag{19.a}$$
$$\overline{\Lambda^{(1)}(\pi,2\pi-\varphi,\lambda)}\hat{\mathbf{q}}^{[1]} = [(\overline{\mathbf{B}^{(1)}}^{\mathrm{T}}\mathbf{A}^{(2)}+\overline{\mathbf{A}^{(1)}}^{\mathrm{T}}\mathbf{B}^{(2)})\Lambda^{(2)}(\pi,2\pi-\varphi,\lambda)$$
$$+(\overline{\mathbf{B}^{(1)}}^{\mathrm{T}}\overline{\mathbf{A}^{(2)}}+\overline{\mathbf{A}^{(1)}}^{\mathrm{T}}\overline{\mathbf{B}^{(2)}})\overline{\Lambda^{(2)}(\pi,2\pi-\varphi,\bar{\lambda})}]\hat{\mathbf{q}}^{[2]} \tag{19.b}$$

where $\hat{\mathbf{q}}^{[1]} = \hat{\mathbf{q}}_1^{[1]}+\hat{\mathbf{q}}_2^{[1]}$. Substituting $\Lambda^{(1)}$ and $\Lambda^{(2)}$ into these two equations, one obtains:

$$\begin{bmatrix} \mathbf{I} & -e^{-i\lambda\pi}\mathbf{I}-\mathbf{B}^{(2)}\langle(\zeta^{(2)}(-\varphi))^{\lambda}\rangle\mathbf{B}^{(2)^{-1}}\mathbf{M}^{22}\overline{\mathbf{B}^{(2)}}\langle(-\overline{\zeta^{(2)}(-\varphi)})^{-\lambda}\rangle\overline{\mathbf{B}^{(2)}}^{-1} \\ \mathbf{I} & \overline{\mathbf{B}^{(2)}}\langle(\overline{\zeta^{(2)}(-\varphi)})^{\lambda}\rangle\overline{\mathbf{B}^{(2)}}^{-1}\overline{\mathbf{M}^{22}}\mathbf{B}^{(2)}\langle(-\zeta^{(2)}(-\varphi))^{-\lambda}\rangle\mathbf{B}^{(2)^{-1}}+e^{i\lambda\pi}\mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{q}}^{[1]} \\ \mathbf{B}^{(2)}\mathbf{B}^{(2)^{\mathrm{T}}}\hat{\mathbf{q}}^{[2]} \end{bmatrix} = 0 \tag{20}$$

where

$$\mathbf{M}^{22} = -\overline{\mathbf{F}^{-1}\mathbf{G}} \tag{21}$$

and

$$\mathbf{F} = (i\mathbf{A}^{(1)}\mathbf{B}^{(1)^{-1}})+\overline{(i\mathbf{A}^{(2)}\mathbf{B}^{(2)^{-1}})} \tag{22}$$

$$\mathbf{G} = (i\mathbf{A}^{(1)}\mathbf{B}^{(1)^{-1}})-(i\mathbf{A}^{(2)}\mathbf{B}^{(2)^{-1}}). \tag{23}$$

For nontrivial solutions of $\hat{\mathbf{q}}^{[1]}$ and $\hat{\mathbf{q}}^{[2]}$ of eqn (20), one leads to the following characteristic equation:

$$\det[\cos(\pi\lambda)\mathbf{I}+\mathbf{Q}(\lambda)] = 0, \quad 0 < \mathrm{Re}\,\{\lambda\} < 1 \tag{24}$$

where

$$\mathbf{Q}(\lambda) = \frac{1}{2}\sum_{k=1}^{3}\sum_{j=1}^{3}\left[\left(\frac{\cos\varphi-p_k^{(2)}\sin\varphi}{-(\cos\varphi-\overline{p_k^{(2)}}\sin\varphi)}\right)^{\lambda}\mathbf{E}_k^{(2)}\mathbf{M}^{22}\overline{\mathbf{E}_j^{(2)}} + \left(\frac{\cos\varphi-\overline{p_k^{(2)}}\sin\varphi}{-(\cos\varphi-p_k^{(2)}\sin\varphi)}\right)^{\lambda}\overline{\mathbf{E}_k^{(2)}\mathbf{M}^{22}\mathbf{E}_j^{(2)}}\right] \tag{25}$$

$$\mathbf{E}_k^{(\alpha)} = \mathbf{B}^{(\alpha)}\mathbf{I}_k\mathbf{B}^{(\alpha)^{-1}}, \quad \alpha = 1, 2. \tag{26}$$

Apparently, if λ is a root of eqn (24), then $\bar{\lambda}$ is also a root of that equation. It is noted that when the crack terminates normally at the interface, i.e., when $\varphi = 90°$, eqn (24) reduces to that obtained by Sung and Liou (1996). Also note that for the special cases when $\varphi = 0°$ or π, quantity $\mathbf{Q}(\lambda)$ defined in eqn (25) becomes

$$\mathbf{Q}(\lambda) = \tfrac{1}{2}(e^{i\pi\lambda}\mathbf{M}^{22} + e^{-i\pi\lambda}\overline{\mathbf{M}^{22}}). \tag{27}$$

Hence, the characteristic roots λ for the interfacial crack problem are

$$\lambda = \frac{1}{2} \pm i\frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta} \tag{28}$$

where $\beta = [-\frac{1}{2}tr(\mathbf{W}\mathbf{D}^{-1})^2]^{1/2}$ and $\mathbf{D} = \text{Re}(\mathbf{F})$ and $\mathbf{W} = \text{Im}(\mathbf{F})$ (Ting (1986)).

4. SINGULARITIES FOR ORTHOTROPIC BIMATERIALS

Characteristic eqn (24) derived in the previous section is usually in complex form, provided that media on both sides of the interface are generally anisotropic. Hence, it is usually difficult to investigate the general features of the characteristic roots for such anisotropic media. We will in the following focus on materials composed by orthotropic media for which the characteristic equation can be expressed in terms of material constants in an explicit form.

The problem considered is sketched in Fig. 1 where the principal axes of material #2 are aligned along the coordinate axes while those of material #1 can have an arbitrary angle γ relative to the interface boundary. Since both of the materials' principal axes in the out-of-plane direction are assumed to be parallel with the x_3-axis, the anti-plane deformation will be decoupled from the in-plane deformation. Hence, the anti-plane deformation is ignored in the following. Due to this fact, the size of all matrices and vectors previously defined and appearing in what follows will be 2×2 and 2×1, respectively. As has been discussed in the paper (Sung and Liou, 1996), the matrix $\mathbf{E}_k^{(2)}$ and quantity $p_k^{(2)}$ appearing in (25) can be expressed in terms of two Krenk's parameters, $\delta^{(2)}$ and $\kappa^{(2)}$ (Krenk, 1979; Sung and Liou, 1996) as follows

$$\mathbf{E}_1^{(2)} = \begin{cases} \dfrac{1}{2\omega_-^{(2)}}\begin{bmatrix} \omega_-^{(2)} + \omega_+^{(2)} & i\delta^{(2)} \\ i\delta^{(2)^{-1}} & \omega_-^{(2)} - \omega_+^{(2)} \end{bmatrix}, & \kappa^{(2)} > 1 \\ \dfrac{1}{2\omega_-^{(2)}}\begin{bmatrix} \omega_-^{(2)} - i\omega_+^{(2)} & \delta^{(2)} \\ \delta^{(2)^{-1}} & \omega_-^{(2)} + i\omega_+^{(2)} \end{bmatrix}, & |\kappa^{(2)}| < 1 \end{cases}$$

$$\mathbf{E}_2^{(2)} = \begin{cases} \dfrac{1}{2\omega_-^{(2)}}\begin{bmatrix} \omega_-^{(2)} - \omega_+^{(2)} & -i\delta^{(2)} \\ -i\delta^{(2)^{-1}} & \omega_-^{(2)} + \omega_+^{(2)} \end{bmatrix}, & \kappa^{(2)} > 1 \\ \dfrac{1}{2\omega_-^{(2)}}\begin{bmatrix} \omega_-^{(2)} + i\omega_+^{(2)} & -\delta^{(2)} \\ -\delta^{(2)^{-1}} & \omega_-^{(2)} - i\omega_+^{(2)} \end{bmatrix}, & |\kappa^{(2)}| < 1 \end{cases} \tag{29}$$

$$p_1^{(2)} = \begin{cases} i\delta^{(2)}(\omega_+^{(2)} + \omega_-^{(2)}), & \kappa^{(2)} > 1 \\ \delta^{(2)}(i\omega_+^{(2)} - \omega_-^{(2)}), & |\kappa^{(2)}| < 1 \end{cases}$$

$$p_2^{(2)} = \begin{cases} i\delta^{(2)}(\omega_+^{(2)} - \omega_-^{(2)}), & \kappa^{(2)} > 1 \\ \delta^{(2)}(i\omega_+^{(2)} + \omega_-^{(2)}), & |\kappa^{(2)}| < 1 \end{cases} \tag{30}$$

where

$$\omega_+^{(\alpha)} = \sqrt{(1+\kappa^{(\alpha)})/2}, \quad \omega_-^{(\alpha)} = \sqrt{|1-\kappa^{(\alpha)}|/2}, \quad \alpha = 1,2. \tag{31}$$

Matrix $\mathbf{M}^{22}$ in (21) can also be expressed in terms of six generalized Dundurs' constants as (Sung and Liou, 1996)

$$\mathbf{M}^{22} = \frac{1}{1-\beta_1\beta_2-\lambda_1\lambda_2}\begin{bmatrix} \alpha_1+\beta_1\beta_2+\lambda_1\lambda_2 & (\lambda_1-i\beta_1)(1+\alpha_2) \\ (\lambda_2+i\beta_2)(1+\alpha_1) & \alpha_2+\beta_1\beta_2+\lambda_1\lambda_2 \end{bmatrix}. \tag{32}$$

The six generalized Dundurs' constants are defined as

$$\begin{pmatrix}\alpha_1\\ \beta_1\\ \lambda_1\end{pmatrix} = \frac{1}{(1+\alpha_0)+d\Delta^{-1}}\begin{pmatrix}(1+\alpha_0)-d\Delta^{-1}\\ \delta^{(2)}2\beta_0\\ \delta^{(2)}d^{**}\end{pmatrix}$$

$$\begin{pmatrix}\alpha_2\\ \beta_2\\ \lambda_2\end{pmatrix} = \frac{1}{(1+\alpha_0)+d^*\Delta}\begin{pmatrix}(1+\alpha_0)-d^*\Delta\\ \delta^{(2)^{-1}}2\beta_0\\ \delta^{(2)^{-1}}d^{**}\end{pmatrix} \tag{33}$$

where

$$\alpha_0 = \frac{\dfrac{\omega_+^{(2)}}{E^{(2)}} - \dfrac{\omega_+^{(1)}}{E^{(1)}}}{\dfrac{\omega_+^{(2)}}{E^{(2)}} + \dfrac{\omega_+^{(1)}}{E^{(1)}}} \quad \beta_0 = \frac{\dfrac{1-\nu^{(2)}}{E^{(2)}} - \dfrac{1-\nu^{(1)}}{E^{(1)}}}{2\left(\dfrac{\omega_+^{(2)}}{E^{(2)}} + \dfrac{\omega_+^{(1)}}{E^{(1)}}\right)} \tag{34.a}$$

and

$$\begin{aligned} d &= (1-\alpha_0)\,\Delta^{-1}(1+\sin^2(\gamma)(\delta^{(1)^2}-1)) \\ d^* &= (1-\alpha_0)\,\Delta(1+\sin^2(\gamma)(\delta^{(1)^{-2}}-1)) \\ d^{**} &= (1-\alpha_0)\sin(\gamma)\cos(\gamma)(\delta^{(1)}-\delta^{(1)^{-1}}) \\ \Delta &= \delta^{(1)}/\delta^{(2)}. \end{aligned} \tag{34.b}$$

Material parameters $E^{(\alpha)}$ and $\nu^{(\alpha)}$ appearing above are the other two Krenk's parameters. These four Krenk's parameters $E^{(\alpha)}$, $\nu^{(\alpha)}$, $\delta^{(\alpha)}$, and $\kappa^{(\alpha)}$ are related to engineering elastic constants. For details please refer to the paper by Krenk (1979). For isotropic materials, Krenk's parameters take the following special values:

$$\begin{aligned} \kappa^{(\alpha)} &= 1 \\ \delta^{(\alpha)} &= 1 \\ E^{(\alpha)} &= E_i^{(\alpha)}/(1-\nu_i^{(\alpha)^2}) \\ \nu^{(\alpha)} &= \nu_i^{(\alpha)}/(1-\nu_i^{(\alpha)}) \quad \text{(plane strain case)} \end{aligned} \tag{35}$$

where $E_i^{(\alpha)}$ is the Young's modulus and $\nu_i^{(\alpha)}$ is the Poisson ratio of the isotropic material (subscript i is not a free index here). The six generalized Dundurs' constants will become, for isotropic material,

$$\begin{aligned} \alpha_1 &= \alpha_2 = \alpha_0 \\ \beta_1 &= \beta_2 = \beta_0 \\ \lambda_1 &= \lambda_2 = 0 \end{aligned} \tag{36}$$

where α_0 and β_0 are reduced, respectively, to the Dundurs' constants α and β (1968).

Substituting results of (29), (30) and (32) into (25), one can obtain the explicit expressions for the elements of $\mathbf{Q}$ for orthotropic bimaterials which are shown in Appendix A. By letting $\varphi = \pi/2$, the explicit forms of the elements of $\mathbf{Q}(\lambda)$ shown in Appendix A are reduced to those developed by Sung and Liou (1996). Also note that when $\delta^{(1)} = 1$, the generalized Dundurs' constants defined by eqn (33) are all independent of γ (i.e., independent of the alignments of the upper material's principal axes), and moreover, γ appears nowhere in other quantities, therefore all elements of matrix $\mathbf{Q}$ will be independent of γ. This implies that the alignments of material #1 will have no effect on the characteristic roots λ when material #1 has the special property that $\delta^{(1)} = 1$. Such phenomena has also been observed by Sung and Liou (1996). Let us now consider the interfacial crack problem, i.e., $\varphi = 0$ or π. For these cases, one can easily show that elements of $\mathbf{Q}$ are:

$$\begin{aligned} \mathbf{Q}(1,1) &= N_{11}\cos(\lambda\pi) \\ \mathbf{Q}(2,2) &= N_{22}\cos(\lambda\pi) \\ \mathbf{Q}(1,2) &= \delta^{(2)}(N_{12}\sin(\lambda\pi)+\hat{N}_{12}\cos(\lambda\pi)) \\ \mathbf{Q}(2,1) &= \delta^{(2)^{-1}}(N_{21}\sin(\lambda\pi)+\hat{N}_{21}\cos(\lambda\pi)). \end{aligned} \tag{37}$$

Hence, with the definition of $\hat{N}_{12}$, $\hat{N}_{21}$, and N_{ij} expressed in (A-3), the characteristic equation becomes

$$\cos(\lambda\pi)^2-(\sin(\lambda\pi)\beta_1+\cos(\lambda\pi)\lambda_1)(-\sin(\lambda\pi)\beta_2+\cos(\lambda\pi)\lambda_2) = 0. \tag{38}$$

Therefore,

$$\lambda = \tfrac{1}{2}\pm i\varepsilon \tag{39}$$

where oscillatory index ε is given by

$$\varepsilon = \frac{1}{2\pi}\ln\frac{1+\sqrt{\beta_1\beta_2/(1-\lambda_1\lambda_2)}}{1-\sqrt{\beta_1\beta_2/(1-\lambda_1\lambda_2)}}. \tag{40}$$

It is seen that, for the orthotropic bimaterial problem, ε is determined only by parameters β_1, β_2, λ_1 and λ_2. It is not related to α_1 and α_2 at all. When material #1 is aligned, i.e., $\gamma = 0°$, ε will be determined only by two generalized Dundurs' constants, β_1 and β_2 (since $\lambda_1 = \lambda_2 = 0$ when $\gamma = 0°$). When material #1 has the special property that $\delta^{(1)} = 1$ then, even though $\gamma \neq 0°$, ε will be also related to β_1 and β_2 only (since $\lambda_1 = \lambda_2 = 0$ when $\delta^{(1)} = 1$). When both materials are isotropic, then parameters in eqn (40) are $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = 0$. Hence, the oscillatory index ε becomes the well known result:

$$\varepsilon = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}. \tag{41}$$

It is interesting to consider the conditions for which the oscillatory behavior will disappear.

From eqn (40), we see that these conditions are $\beta_1 = 0$ or $\beta_2 = 0$. According to the definition of generalized Dundurs' constants (see eqn (33)), these two conditions are equivalent to

$$\beta_0 = 0 \tag{42}$$

or, from eqn (34.a), to

$$\frac{1-\nu^{(2)}}{E^{(2)}} = \frac{1-\nu^{(1)}}{E^{(1)}}. \tag{43}$$

Note that the condition of $\beta_0 = 0$ implies $\mathbf{W} \equiv 0$. The vanishing of $\mathbf{W}$ for the absence of oscillating behavior for general anisotropic interfacial problems has been observed by Qu and Bassani (1993). Here we give the conditions for orthotropic media in terms of material parameter β_0.

For the degenerate materials, the present form of matrix $\mathbf{Q}$ expressed in Appendix A can not be directly employed since the denominators of the elements of $\mathbf{Q}$ contain the term of $(1-\kappa^{(2)})$ which will vanish when material #2 becomes degenerate, i.e., $\kappa^{(2)} = 1$. However, through the L'Hospital rule, the explicit expressions for the elements of $\mathbf{Q}$ are also available which are listed in Appendix B. They are valid as long as $\kappa^{(2)} = 1$. Many other degenerate cases can be further studied from these results. For instance, letting $\delta^{(2)} = 1$ in those expressions shown in Appendix B, then the results can be employed to study the singularities of an isotropic medium joined to an orthotropic medium. By letting $\delta^{(1)} = \kappa^{(1)} = 1$ and $\delta^{(2)} = 1$, then the problem which has been studied by Bogy (1971) becomes two isotropic materials with different properties joined together. The characteristic equation set up by Bogy (1971) can be recovered from our expressions, just noting that

$$\begin{aligned} \hat{N}_{12} &= \hat{N}_{21} = 0 \\ N_{11} &= N_{22} = \frac{\alpha+\beta^2}{1-\beta^2} \\ N_{12} &= N_{21} = \frac{\beta(1+\alpha)}{1-\beta^2} \end{aligned} \tag{44}$$

when $\delta^{(1)} = \kappa^{(1)} = 1$ and $\delta^{(2)} = \kappa^{(2)} = 1$ and substituting these values into eqns (B-1)–(B-4), one can reach the results obtained by Bogy.

5. SOME NUMERICAL RESULTS AND DISCUSSIONS

In this section, the singularities of several kinds of material compositions are analyzed. First we consider the problem of an isotropic medium joined to an orthotropic medium, then the mismatch problem is focused. Finally, a graphite–epoxy composite is considered.

(I) An isotropic medium joined to an orthotropic medium.

We consider first the case of $\gamma = 0°$ which occurred frequently in engineering applications. It is noted that quantities $\hat{N}_{12}$, $\hat{N}_{21}$ and N_{kj} $(k,j = 1,2)$ defined in eqn (A-3) depend on parameters α_0, β_0 and Δ only when $\gamma = 0°$. Therefore, the explicit dependence of the roots λ on the material parameters becomes simpler. For an isotropic medium (#2) joined to an orthotropic medium (#1), one can further let $\delta^{(2)} = \kappa^{(2)} = 1$. Hence, the dependence of the roots λ on the material parameters becomes

$$\lambda = \lambda(\alpha_0, \beta_0, \Delta). \tag{45}$$

Only three material parameters α_0, β_0 and Δ $(=\delta^{(1)})$ will enter into following discussions. As has been discussed in the paper by Sung and Liou (1996), the possible values of α_0 and

β_0 fall in a region enclosed by a quadrilateral in a α_0–β_0 diagram. The ranges of α_0 and β_0 are $-1 \leqslant \alpha_0 \leqslant 1$ and $(-1/2\omega_+^{(1)}) \leqslant \beta_0 \leqslant (1/2\omega_+^{(2)})$, respectively, if the nonnegative values of $v^{(\alpha)}$ are assumed.

Taking that assumption for granted and taking $\beta_0 = m\alpha_0(|\alpha_0| \leqslant 1)$ for the present analysis where $m = 0$ or 0.25, Fig. 2a–c present the roots λ vs α_0 for $\varphi = \pi/6$, $\pi/4$ and $\pi/3$, respectively, for $m = 0$, while those in Fig. 3a–c are the results of λ for $m = 0.25$. In each figure, several values of Δ are chosen for analysis. It is observed from Fig. 2a–c that for each Δ there corresponds to two real roots for $|\alpha_0| < 1$ except at the point $(\alpha_0, \beta_0) = (0,0)$ with $\Delta = 1$ where only one real root is observed. At this special value of material combinations, i.e., $(\alpha_0, \beta_0) = (0,0)$ and $\Delta = 1$, the elements of $\mathbf{Q}$ are

$$\mathbf{Q}(1,1) = \mathbf{Q}(2,2) = \mathbf{Q}(1,2) = \mathbf{Q}(2,1) = 0. \tag{46}$$

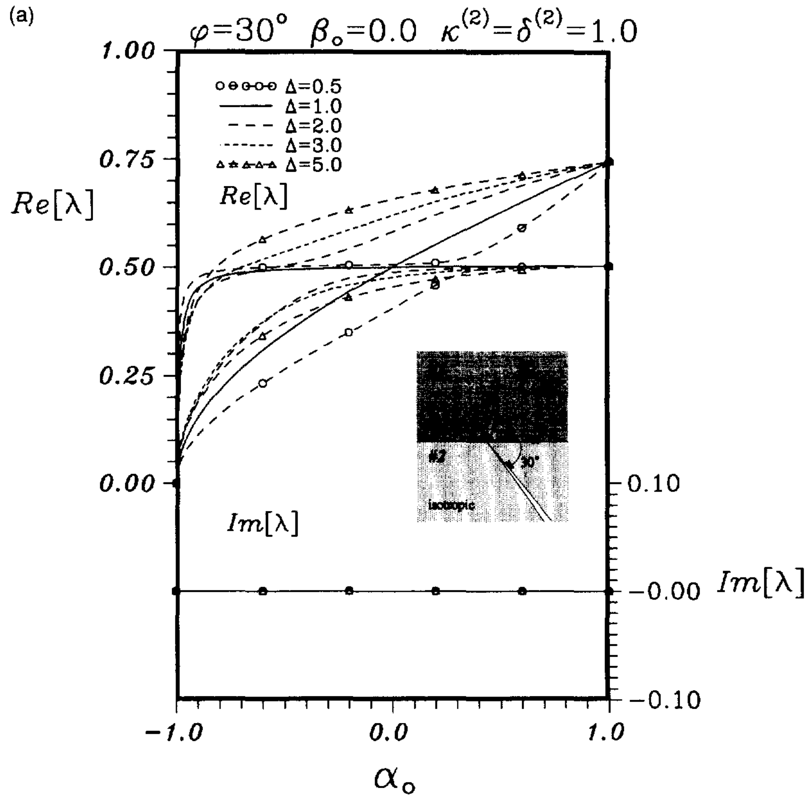
Hence, only one real root $\lambda = 1/2$ is obtained. As α_0 approaches 1, the roots are found to be independent of Δ while as α_0 approaches -1, all roots become $\lambda = 0$ which are again independent of Δ. These phenomena which can be verified directly from the characteristic equation given by eqn (24) have also been observed in the investigations of a crack terminating normally at the interface (Sung and Liou, 1996). Complex roots appearing in complex conjugate form are observed for certain values of α_0 as shown in Fig. 3a and 3b with $\beta_0 = 0.25\alpha_0$. The phenomena of λ previously noted at two extremities of α_0, i.e., $\alpha_0 = -1$ and $\alpha_0 = 1$, are also observed for the case of $\beta_0 = 0.25\alpha_0$. It should be mentioned that those results corresponding to $\Delta = 1$ plotted in Fig. 2 and Fig. 3 are applicable for isotropic bimaterial problems if their properties characterized by Dundurs' constants are such that $\alpha = \alpha_0$ and $\beta = \beta_0$.

It is noted that for isotropic bimaterial problems, parameters α_0 and β_0 reduce to Dundurs' constants α and β, respectively. These two parameters are the measure of the elastic dissimilarity of two isotropic materials. Parameter α can be further interpreted as a measure of the dissimilarity in stiffness of the two materials, which has been noted by Suo (1989). When $\alpha > 0$, the material #1 is stiffer than #2 while has $\alpha < 0$ the material #1 is relatively compliant. However, for orthotropic bimaterial problems the behaviors at the interface are strongly influenced by material alignments. To select a parameter (or parameters) to characterize which material is stiffer is more complicated. The parameter α_0, which reduces to α for two isotropic materials, will be a guess to be a candidate in the measure of the dissimilarity. Our results shown in Fig. 2 show that the values of λ increase as α_0 runs from -1 to 1. This more or less implies that parameter α_0, defined for two orthotropic materials, does play a similar role as α does for isotropic materials. However, there are cases that as α_0 increases the value of λ will decrease, as shown in Fig. 3a and b. These observations reflect that the role of parameter α_0 played in two orthotropic materials is not the same as α played in isotropic materials. Let's further consider the simplest case of a mismatch problem, i.e., the problem composed by materials having the same orthotropic properties joined together at the interface boundary with one of the material's principal axes having an angle γ relative to the interface. For such a problem, it is clear that to select a parameter to measure the dissimilarity in stiffness of the same materials seems meaningless.

For the comparisons of the magnitudes of λ for different meeting angles, we make another two plots shown in Fig. 4a and b for the case of $\beta_0 = 0.0$. Also plotted in these figures are the results of $\varphi = \pi/2$ which have been discussed by Sung and Liou (1996). Only one real root will occur for the case of $\varphi = \pi/2$ for all values of α_0 with $\Delta = 1$. For meeting angles other than $\varphi = \pi/2$, it is seen from Fig. 4(a) that their corresponding dominant singular terms are always higher than those for $\varphi = \pi/2$ for all values of α_0 (except at $\alpha_0 = 0.0$) for the case of $\Delta = 1$. However, for the case of $\Delta = 2$ (Fig. 4(b)), their corresponding dominant singular terms may be smaller than those for $\varphi = \pi/2$.

The above studies are all for the case of $\gamma = 0$. From numerical studies we observed that as long as $\gamma = 0$ and $\beta_0 = 0$ characteristic roots λ seem to be always real for all φ $(0 \leqslant \varphi \leqslant \pi/2)$. However, if $\beta_0 \neq 0$, then complex roots may exist. This observation of λ to be a real value, i.e., the oscillatory behavior at the tip will disappear, will be further noted in the later discussions.

(a)



(b)

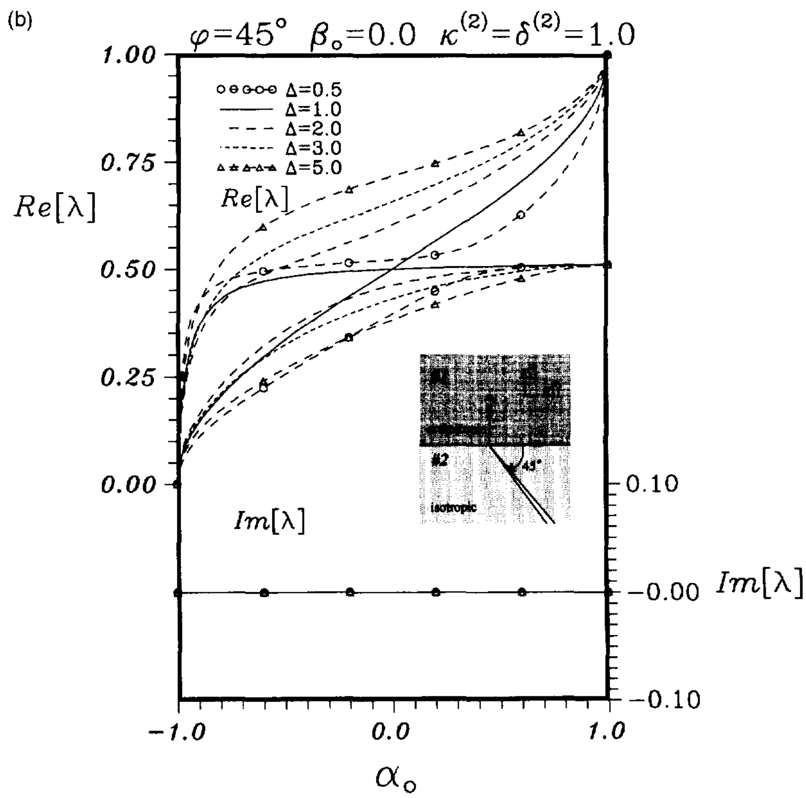


Fig. 2. Characteristic roots λ vs α_0 for (a) $\varphi = \pi/6$, (b) $\varphi = \pi/4$ and (c) $\varphi = \pi/3$ ($\beta_0 = 0$). (*Continued opposite.*)

(c)

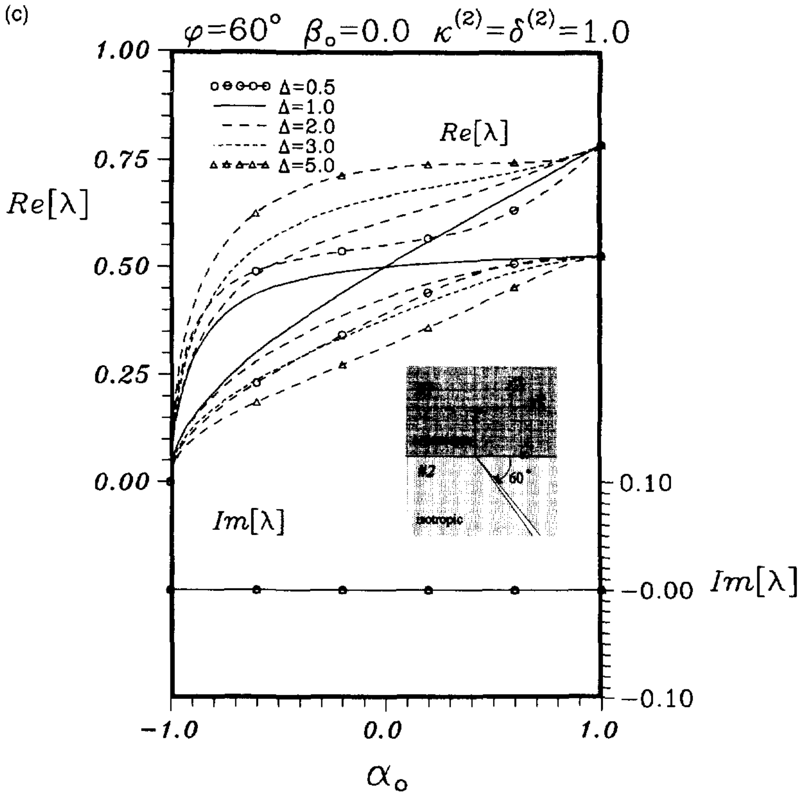


Fig. 2—*Continued.*

Having discussed the problem of material #1 being properly aligned i.e., $\gamma = 0^\circ$, we next present the effect of γ where $0 \leqslant \gamma \leqslant \pi/2$. Properties of material #2 are still kept isotropic, i.e., $\delta^{(2)} = \kappa^{(2)} = 1$. Only the case for which $\beta_0 = 0$ is selected for presentation. As has been discussed previously, the effect of γ on λ will vanish when material #1 has the special property that $\delta^{(1)} = 1$. Hence, the selection of $\Delta = \delta^{(1)}/\delta^{(2)} = \delta^{(1)} = 1$ will have no information about the effect of γ. We therefore take $\Delta = 2$ for presentations. Figure 5a–c are the results of λ vs γ for $\varphi = \pi/6$, $\pi/4$, and $\pi/3$, respectively. The usefulness of these plottings is that for a given orthotropic material one can orient the material principal axes so that the magnitude of the dominant singularity at the interface can be a minimum. Observing these figures, we note that the case of $\alpha_0 = 1.0$ will produce constant λ for all γ. This is because six generalized Dundurs' constants are all independent of γ when $\alpha_0 = 1.0$. It is also noted that complex roots are surely observed in Fig. 5c for certain values of γ. This implies that no guarantee of the absence of the oscillatory behavior when $\gamma \neq 0^\circ$ or π even though the parameter β_0 vanishes. One more thing to be mentioned is that, although the results of λ for Δ and Δ^{-1} are related to each other when $\varphi = \pi/2$ (Sung and Liou, 1996), no such relations hold when $\varphi \neq \pi/2$.

(II) Mismatch problem

For orthotropic bimaterial problems, the investigations of the effect of parameters α_0 on λ would have the same tendency as those previously discussed for the problem of an isotropic material joined to an orthotropic material. Hence, instead of making such an investigation, we focus on the mismatch problem which is composed by two orthotropic materials both having the same properties, but with the material principal axes of #1 have an angle γ relative to the #2 at the interface boundary. For such a mismatch problem, the constants α_0 and β_0 will both vanish and the characteristic equation will depend on the four Krenk's material parameters: $\kappa^{(1)} = \kappa^{(2)} = \kappa$, $\delta^{(1)} = \delta^{(2)} = \delta$, $\nu^{(1)} = \nu^{(2)} = \nu$ and $E^{(1)} = E^{(2)} = E$. Since the influences of E and ν are absorbed by the parameters α_0 and β_0, therefore, only κ and δ need to be considered in the characteristic equation.

Figure 6a–d present the results of λ vs the mismatch angle γ for different meeting angles. The parameter δ ($\delta = 2$) is kept constant. It is seen that at $\gamma = 0$ which corresponds to homogeneous medium the magnitude of the singularity is 1/2, while as the mismatch

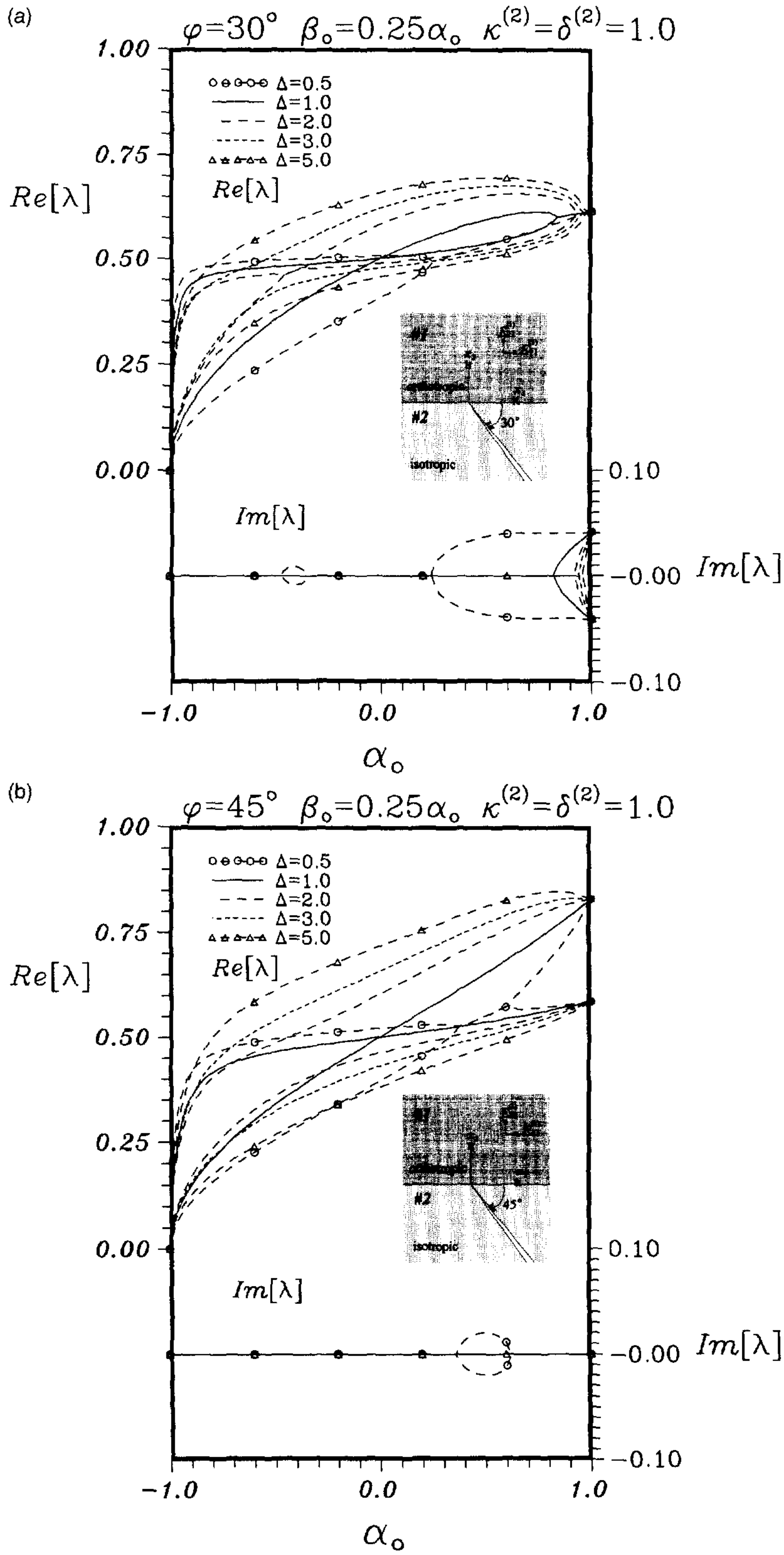


Fig. 3. Characteristic roots λ vs α_0 for (a) $\varphi = \pi/6$, (b) $\varphi = \pi/4$ and (c) $\varphi = \pi/3$ ($\beta_0 = 0.25\alpha_0$).
(*Continued opposite.*)

(c)

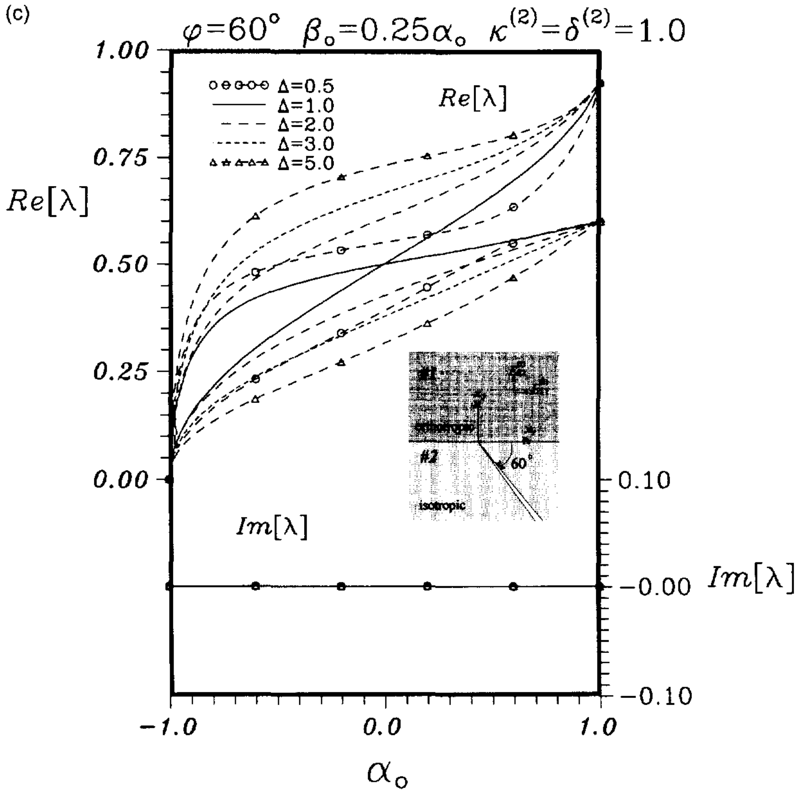


Fig. 3—*Continued.*

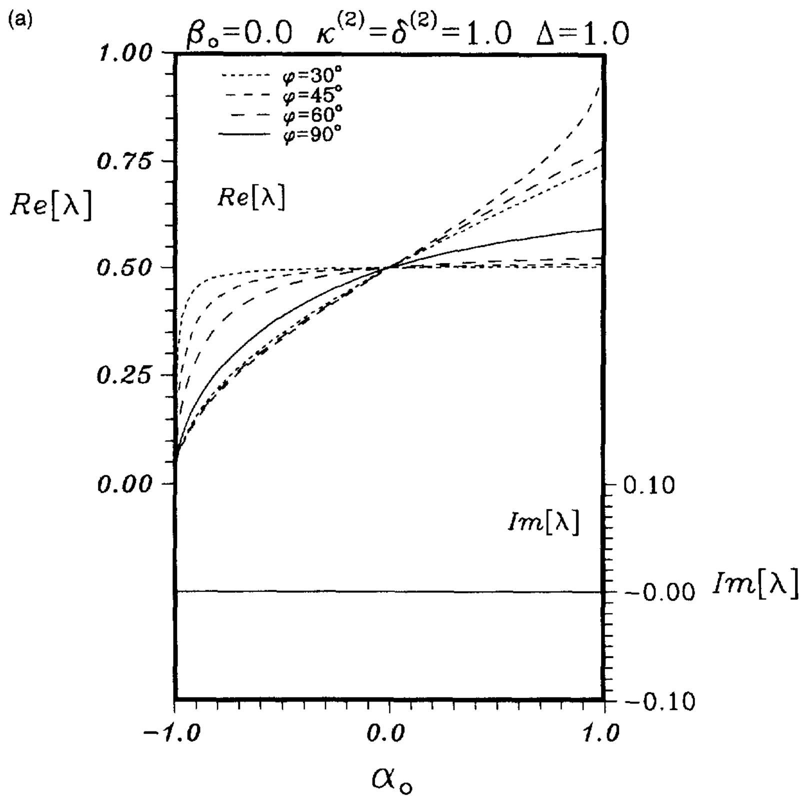
angle γ increases the most dominant singularity $(= -1+\lambda)$ is getting higher. Also observed from these figures is that the effect of parameter κ on λ is small. Similar to those studies of Fig. 6a–d, Fig. 7a–b are the results of λ vs γ but now we kept κ constant $(\kappa = 2)$. Several δ are selected in plotting these figures to see the effect of that parameter. It is observed that as γ increases the most dominant singularity will also increase. Also we note that the effect of parameter δ on λ are more profound than that of the parameter κ. From above analyses we see that as long as mismatch angle γ exists the stress singularity for the present mismatch problem will be higher than that for homogeneous medium. Also we observed from numerical studies that all the roots are real for mismatch problems.

(III) Orthotropic bimaterial problem

The last materials we selected for analysis in layers 1 and 2 are graphite–epoxy composites, since it would be useful to have some information about real particular orthotropic materials that are frequently occurred in engineering applications. For the purpose of comparison, the elastic constants of the materials used by Wu and Erdogan (1993) are adopted here. These constants in units of GPA are

$$\text{Material 1}: E_{1x} = 39.0, \quad E_{1y} = 6.4, \quad E_{1z} = 30.6$$
$$G_{1xy} = 4.5, \quad G_{1yz} = 4.5, \quad G_{1xz} = 19.7$$
$$\nu_{1yx} = 0.275, \quad \nu_{1zy} = 0.275, \quad \nu_{1xz} = 0.447$$
$$\text{Material 2}: E_{2x} = 30.6, \quad E_{2y} = 6.4, \quad E_{2z} = 39.0$$
$$G_{2xy} = 4.5, \quad G_{2yz} = 4.5, \quad G_{2xz} = 19.7$$
$$\nu_{2yx} = 0.275, \quad \nu_{2zy} = 0.275, \quad \nu_{2xz} = 0.351$$

or, in terms of Krenk's parameters, they are



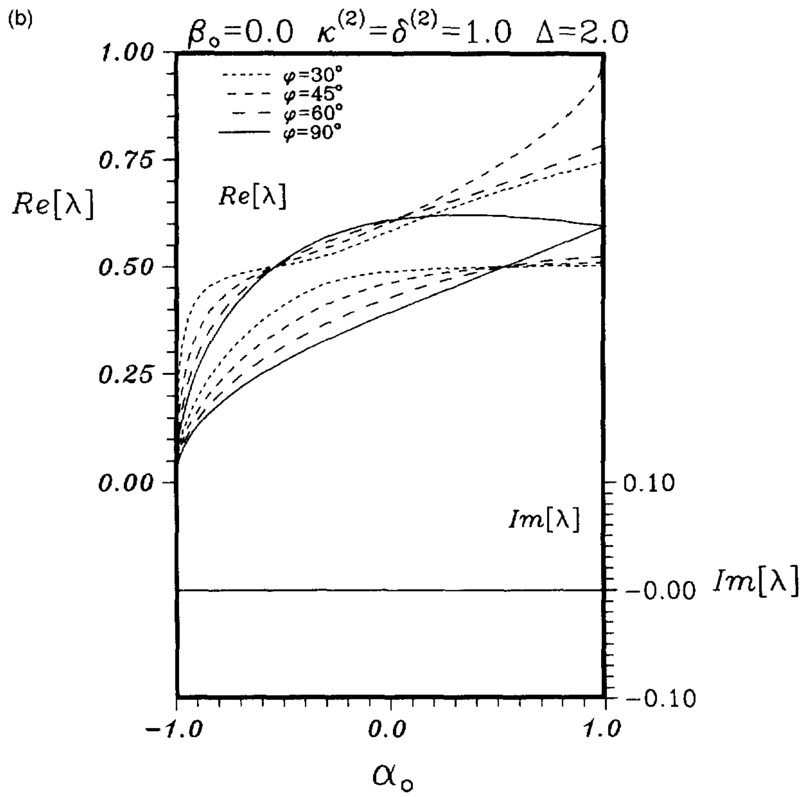
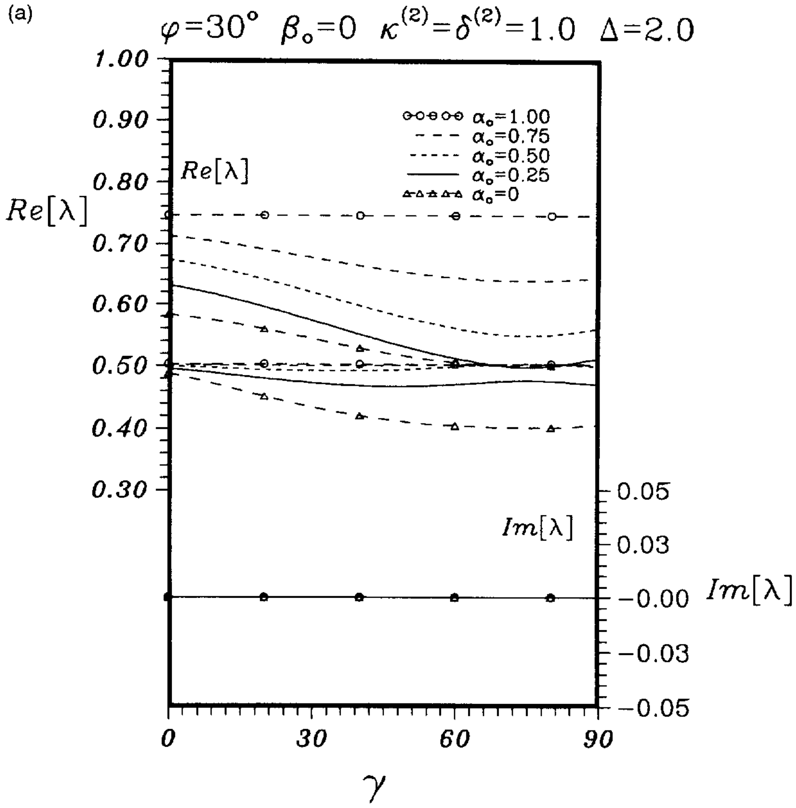


Fig. 4. Comparisons of the roots λ for different meeting angles for (a) $\Delta = 1.0$ and (b) $\Delta = 2.0$.

(a)



(b)

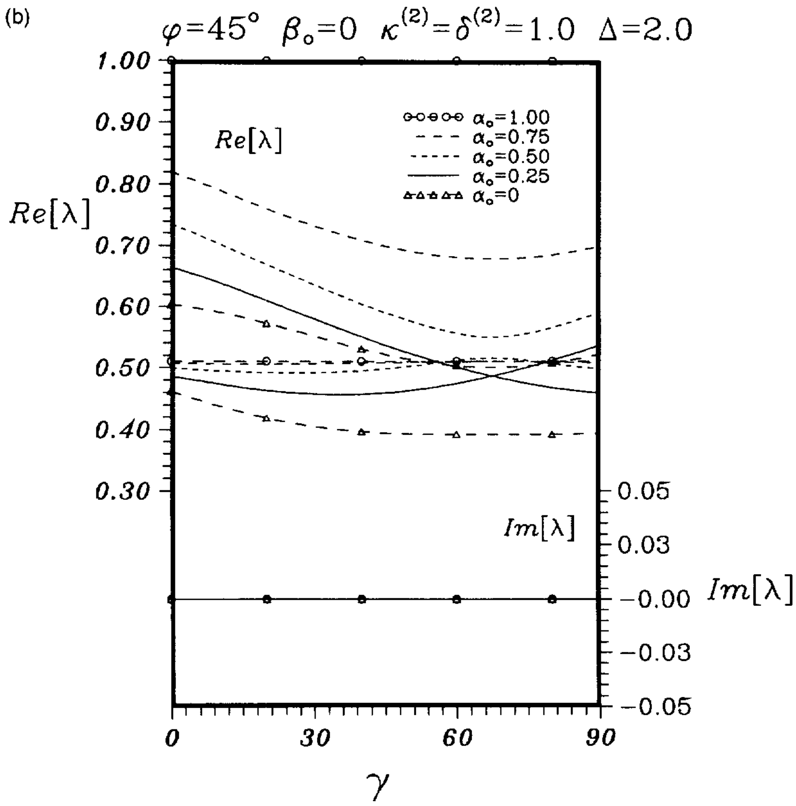


Fig. 5. Characteristic roots λ vs γ for (a) $\varphi = \pi/6$, (b) $\varphi = \pi/4$ and (c) $\varphi = \pi/3$ ($\beta_0 = 0, \Delta = 2.0$).
(*Continued overleaf.*)

(c)

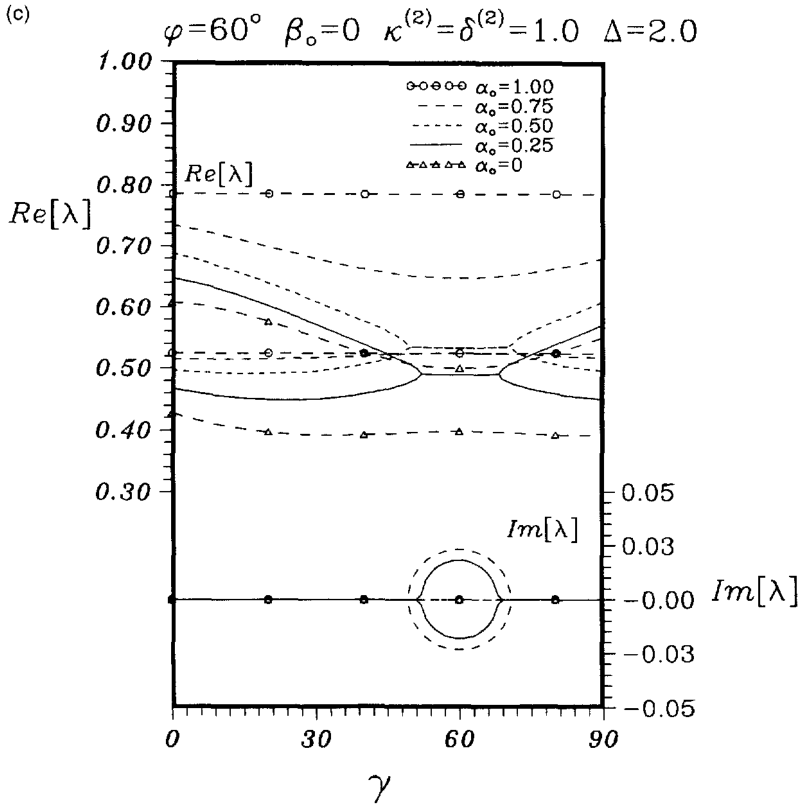


Fig. 5—*Continued.*

$$\text{Material 1}: \delta^{(1)} = 1.4656, \quad \nu^{(1)} = 0.47633, \quad E^{(1)} = 21.533, \quad \kappa^{(1)} = 1.9162$$

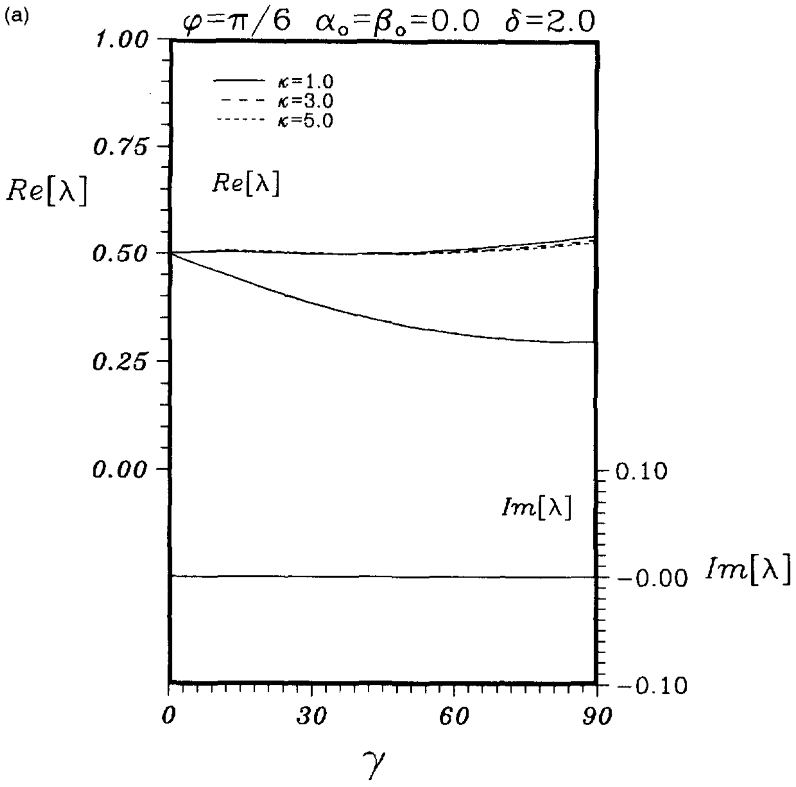
$$\text{Material 2}: \delta^{(2)} = 1.3224, \quad \nu^{(2)} = 0.58556, \quad E^{(2)} = 20.758, \quad \kappa^{(2)} = 1.7209.$$

Two cases are considered below. First we consider the problem of a crack lying in material 1 (called material pair B) and second, the crack lying in material 2 (called material pair A). For material pair A the constants α_0 and β_0 defined in eqn (34.a) are $\alpha_0 = -9.908\times10^{-4}$, $\beta_0 = 1.939\times10^{-2}$ while for material pair B the constants become $\alpha_0 = 9.908\times10^{-4}$, $\beta_0 = -1.939\times10^{-2}$, respectively. Figure 8a–d are the results of λ vs φ for $\gamma = 0$, $\pi/6$, $\pi/3$ and $\pi/2$, respectively. Results of Wu and Erdogan (1993), who investigated only the case of a crack terminating normally ($\varphi = 90°$) for two aligned orthotropic materials ($\gamma = 0$), are also plotted in Fig. 8a. It should be noted that there are two real roots corresponding to each material pair for $\varphi = \pi/2$. Only one of them is presented by Wu and Erdogan (1993). Comparing results for material A and material B, it is seen that the differences of λ due to the crack embedded in either material 1 or material 2 are not so significant since the present materials selected are such that materials 1 and 2 are the same except for a 90° rotation about the y-axis. However, the effect of γ on the singularities is quite significant, as can be seen from these figures.

(IV) Discussions

For the interfacial crack problem, i.e., when $\varphi = 0$ or π, we have shown that the oscillatory behavior will disappear when $\beta_0 = 0$ for orthotropic bimaterial problems. It is curious then to ask what is the condition for an inclined crack (i.e., $\varphi \neq 0$ or π) to ensure the real root always occurs? Our numerical results tend to show that parameter β_0 seems to play an important role. Let's first consider the isotropic bimaterial problems. In this case, constants α_0 and β_0 become α and β, respectively. Let $\beta = 0$ in the characteristic equation given by Bogy (1971). For such a special equation, analytic proving of the fact that it will

(a)



(b)

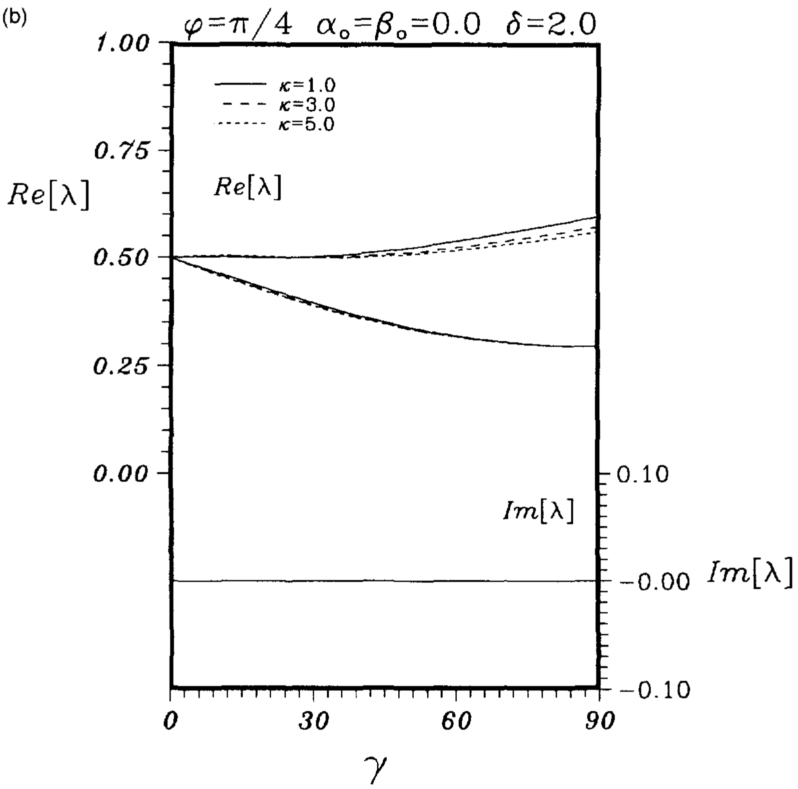
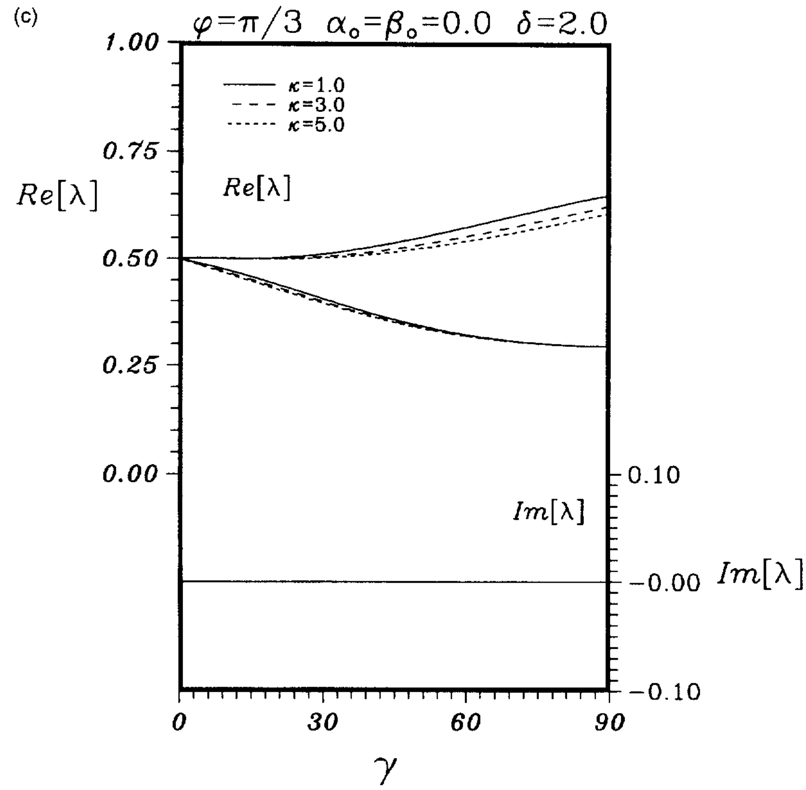


Fig. 6. Characteristic roots λ vs γ for (a) $\varphi = \pi/6$, (b) $\varphi = \pi/4$, (c) $\varphi = \pi/3$ and (d) $\varphi = \pi/2$ (mismatch problem with $\delta = 2$). (*Continued overleaf.*)

(c)



(d)

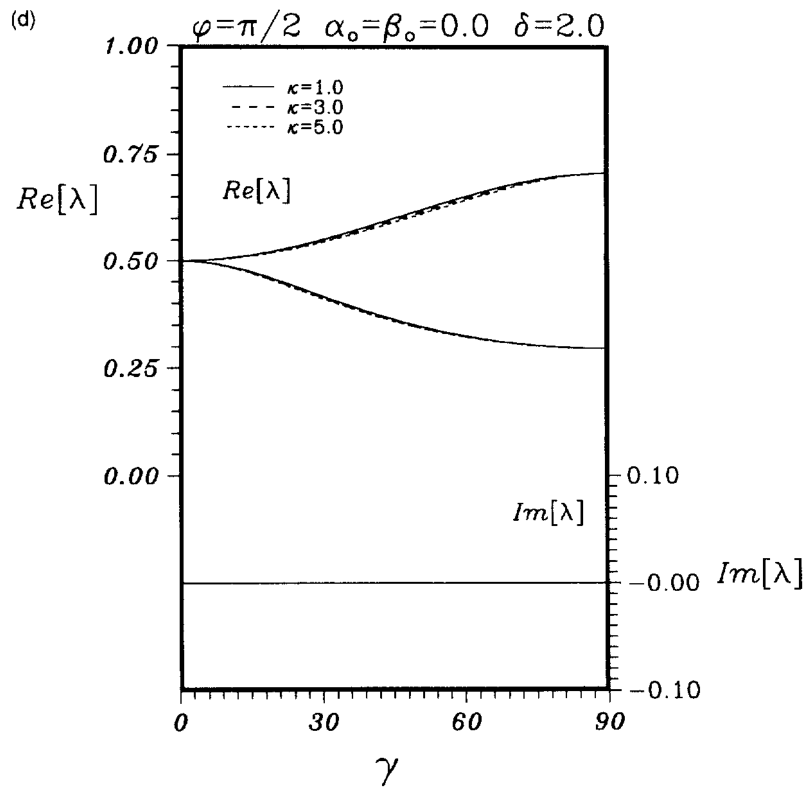
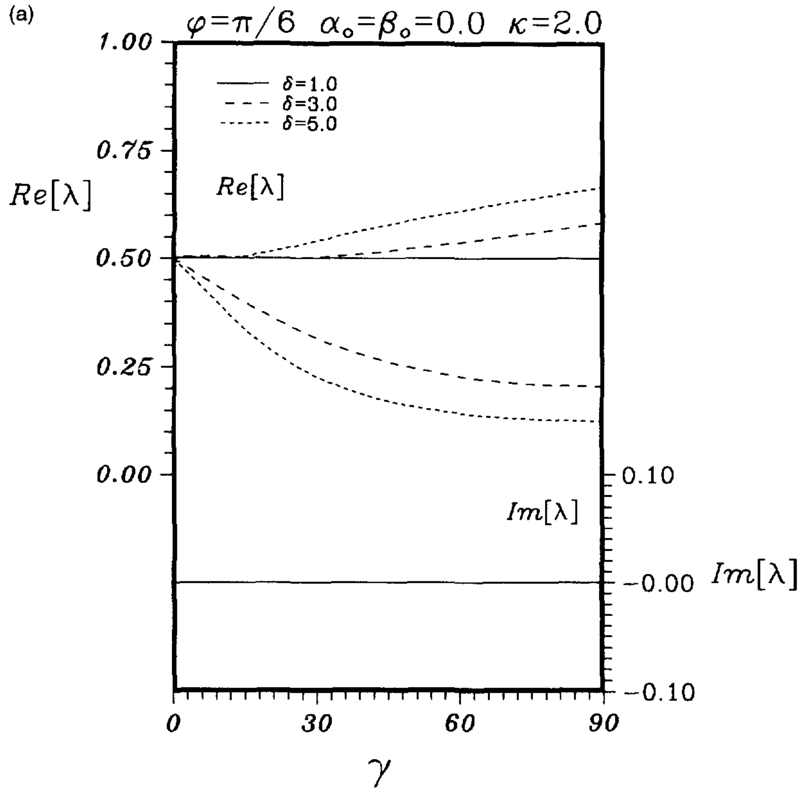


Fig. 6—*Continued.*

(a)



(b)

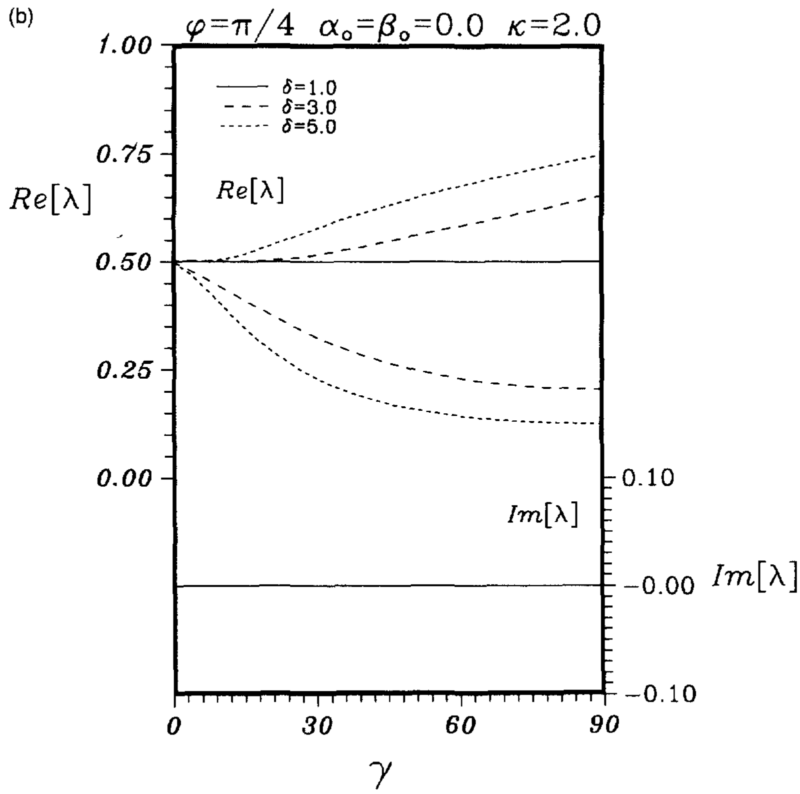
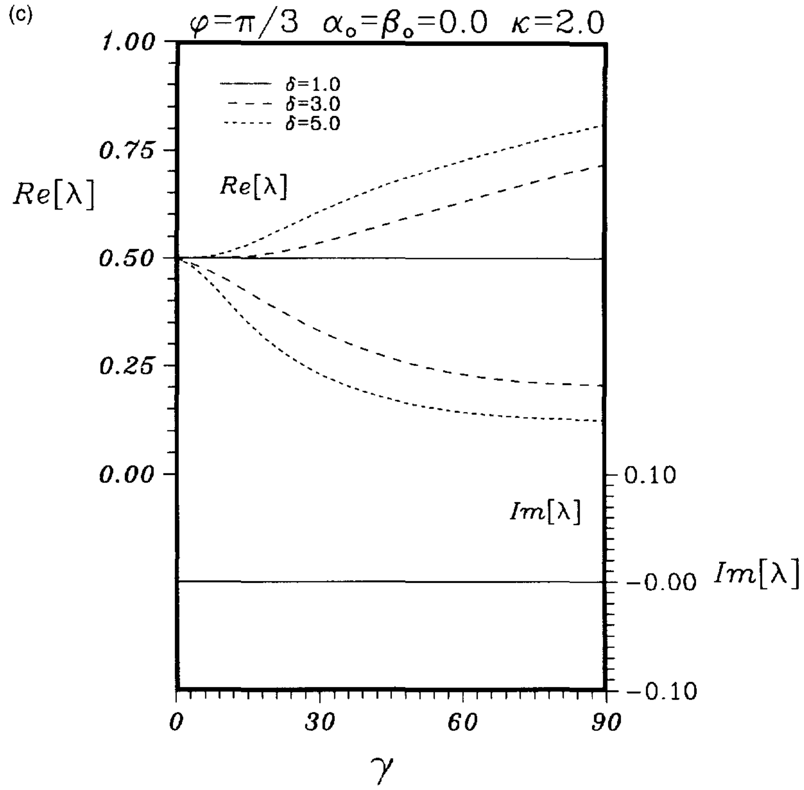


Fig. 7. Characteristic roots λ vs γ for (a) $\varphi = \pi/6$, (b) $\varphi = \pi/4$, (c) $\varphi = \pi/3$ and (d) $\varphi = \pi/2$ (mismatch problem with $\kappa = 2$). (*Continued overleaf.*)

(c)



(d)

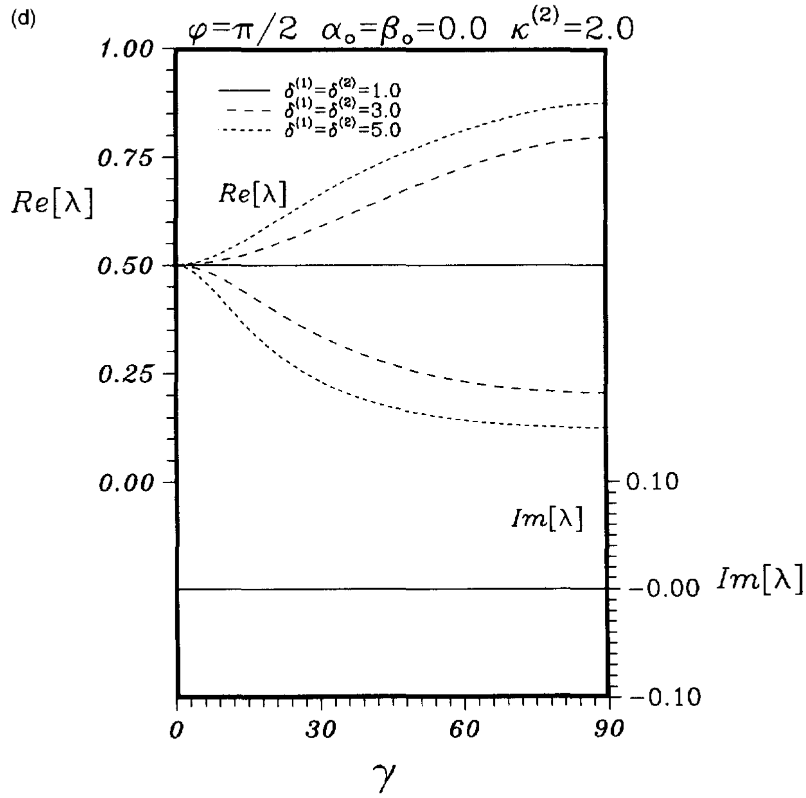
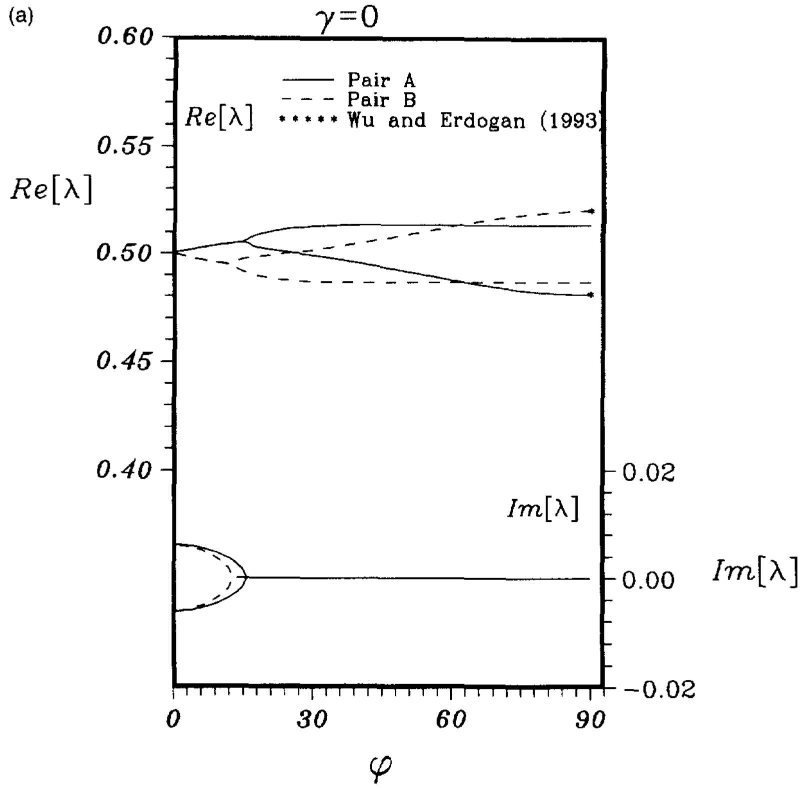


Fig. 7—*Continued.*

(a)



(b)

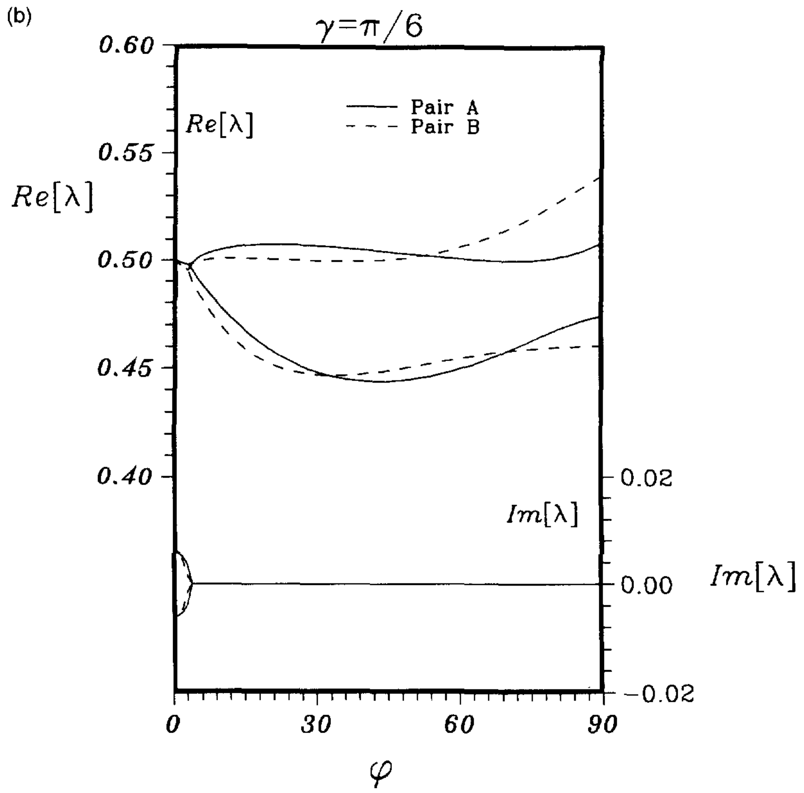
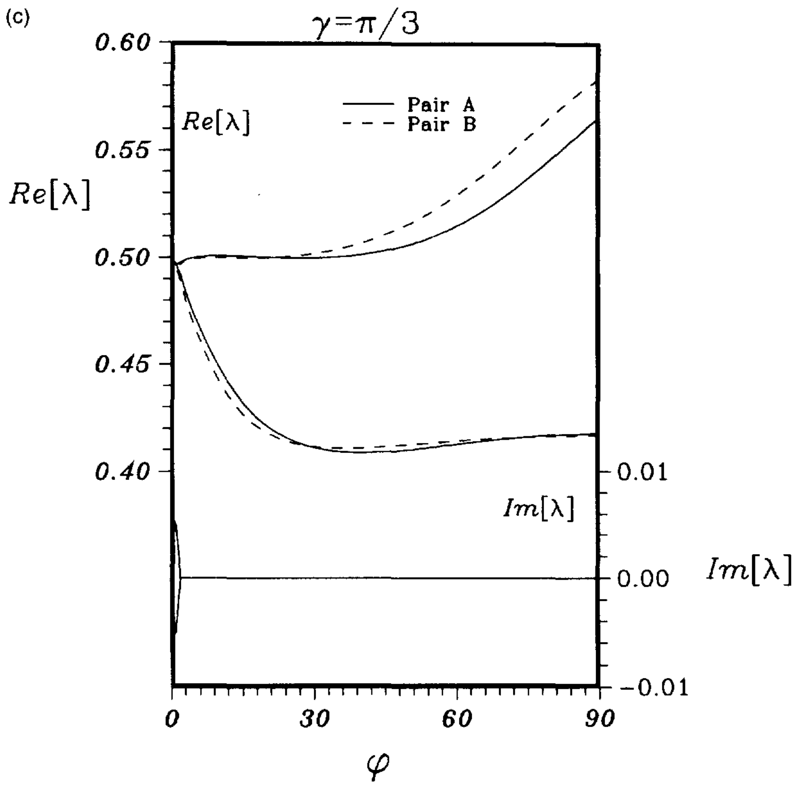


Fig. 8. Characteristic roots λ vs φ for material pairs A and B for (a) $\gamma = 0$, (b) $\gamma = \pi/6$, (c) $\gamma = \pi/3$ and (d) $\gamma = \pi/2$. (*Continued overleaf.*)



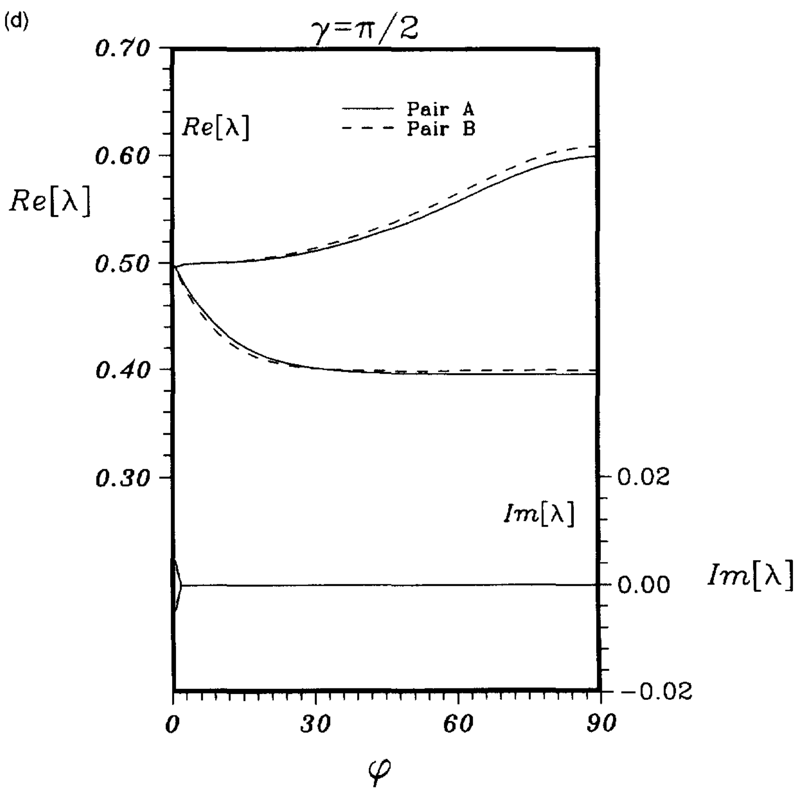


Fig. 8—*Continued.*

always produce real roots λ for a given meeting angle φ for all possible physical values of α is still prohibited. However, Bogy (1971) has presented numerical results of λ for various meeting angles, i.e., for $\varphi = 90^\circ, 110^\circ, 135^\circ, 160^\circ, 175^\circ, 180^\circ$ and for various combinations of bimaterials which are expressed in terms of Dundurs' constants α and β. All his results are plotted in terms of α–β diagram. Observing those figures presented by Bogy (1971), one can find that roots are always real for all values of α, $|\alpha| \leqslant 1$ (nonnegative of Poisson's ratio is considered) whenever $\beta = 0$. If complex roots occur, they occur only at $\beta \neq 0$. These findings are more or less consistent with our previous numerical observations. To get more confidence from numerical points of view, many other computations have been carried out for different material combinations for orthotropic bimaterial problems. All the results, though not presented, show that if orthotropic materials on both sides of interface are properly aligned (i.e., $\gamma = 0$) then real roots λ always occur for all meeting angles as long as $\beta_0 = 0$. According to above discussions, we attempt to make such a statement that for orthotropic bimaterial problems, $\beta_0 = 0$ seems to be the condition of vanishing oscillatory behaviors for a crack having an arbitrary meeting angle provided that both orthotropic media are aligned. Surely this statement needs further confirmation.

6. CONCLUSIONS

A characteristic equation for the stress singularities λ has been derived for an inclined crack terminating at an anisotropic interface. Explicit forms of this equation expressed in terms of material parameters are then given for orthotropic bimaterial problems. Singularities for several types of materials joined at the interface are analyzed and the effects of the material parameters, the meeting angles and the material alignments on the roots are also discussed in some detail. The part that parameter β_0 played in the oscillatory behavior is also noted. Whether the condition of $\beta_0 = 0$ is sufficient to ensure real roots for any meeting angles for two aligned orthotropic materials needs further investigations even though present numerical results reveal this fact.

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APPENDIX A

The explicit expressions for the elements of **Q** for orthotropic bimaterials are, for $\kappa > 1$:

$$
\begin{aligned}
\mathbf{Q}(1,1) = {} & (2\omega_-^{(2)})^{-2}\{N_{11}[\cos(\lambda(\pi-2\varphi_1))(1-2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}) \\
& +\cos(\lambda(\pi-2\varphi_2))(1+2\omega_-^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2})-\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& +N_{22}[\cos(\lambda(\pi-2\varphi_1))+\cos(\lambda(\pi-2\varphi_2))-\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& -N_{12}[\cos(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\cos(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& -N_{21}[\cos(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\cos(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))] \\
& +\hat{N}_{12}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& -\hat{N}_{21}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))]\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}(2,2) = {} & (2\omega_-^{(2)})^{-2}\{N_{11}[\cos(\lambda(\pi-2\varphi_1))+\cos(\lambda(\pi-2\varphi_2)) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& +N_{22}[\cos(\lambda(\pi-2\varphi_1))(1+2\omega_-^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}) \\
& +\cos(\lambda(\pi-2\varphi_2))(1-2\omega_+^{(2)}\omega_-^{(2)}+2\omega_+^{(2)^2})-\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& -N_{12}[\cos(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\cos(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& -N_{21}[\cos(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\cos(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))] \\
& +\hat{N}_{12}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& -\hat{N}_{21}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))]\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}(1,2) = {} & (2\omega_-^{(2)})^{-2}\delta^{(2)}\{N_{11}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& +N_{22}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}+\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}-\omega_-^{(2)}))] \\
& +N_{12}[-\sin(\lambda(\pi-2\varphi_1))-\sin(\lambda(\pi-2\varphi_2)) \\
& +\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(1+2\omega_-^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2})+d^{-1}(1-2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}))] \\
& -N_{21}[\sin(\lambda(\pi-2\varphi_1))+\sin(\lambda(\pi-2\varphi_2))-\sin(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& +\hat{N}_{12}[-\cos(\lambda(\pi-2\varphi_1))-\cos(\lambda(\pi-2\varphi_2)) \\
& +\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(1+2\omega_-^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2})+d^{-1}(1-2\omega_-^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}))] \\
& +\hat{N}_{21}[\cos(\lambda(\pi-2\varphi_1))+\cos(\lambda(\pi-2\varphi_2))-\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})]\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}(2,1) =& (2\omega_-^{(2)})^{-2}\delta^{(2)^{-1}}\{-N_{11}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}-\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}+\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))] \\
& -N_{22}[\sin(\lambda(\pi-2\varphi_1))(\omega_+^{(2)}+\omega_-^{(2)})+\sin(\lambda(\pi-2\varphi_2))(\omega_+^{(2)}-\omega_-^{(2)}) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(\omega_+^{(2)}-\omega_-^{(2)})+d^{-1}(\omega_+^{(2)}+\omega_-^{(2)}))] \\
& +N_{12}[\sin(\lambda(\pi-2\varphi_1))+\sin(\lambda(\pi-2\varphi_2))-\sin(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& +N_{21}[\sin(\lambda(\pi-2\varphi_1))+\sin(\lambda(\pi-2\varphi_2)) \\
& -\sin(\lambda(\pi-\varphi_1-\varphi_2))(d(1-2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2})+d^{-1}(1+2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}))] \\
& +\hat{N}_{12}[\cos(\lambda(\pi-2\varphi_1))+\cos(\lambda(\pi-2\varphi_2))-\cos(\lambda(\pi-\varphi_1-\varphi_2))(d+d^{-1})] \\
& -\hat{N}_{21}[\cos(\lambda(\pi-2\varphi_1))+\cos(\lambda(\pi-2\varphi_2)) \\
& -\cos(\lambda(\pi-\varphi_1-\varphi_2))(d(1-2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2})+d^{-1}(1+2\omega_+^{(2)}\omega_-^{(2)}+2\omega_-^{(2)^2}))]\}
\end{aligned}
\quad \text{(A1)}
$$

where

$$
\begin{aligned}
\varphi_1 &= \tan^{-1}(\delta^{(2)}(\omega_+^{(2)}-\omega_-^{(2)})\tan\varphi) \quad 0 \leqslant \varphi_1 \leqslant \pi \\
\varphi_2 &= \tan^{-1}(\delta^{(2)}(\omega_+^{(2)}+\omega_-^{(2)})\tan\varphi) \quad 0 \leqslant \varphi_2 \leqslant \pi \\
d &= \left(\frac{\sqrt{\cos^2\varphi+(\delta^{(2)}(\omega_+^{(2)}+\omega_-^{(2)})\sin\varphi)^2}}{\sqrt{\cos^2\varphi+(\delta^{(2)}(\omega_+^{(2)}-\omega_-^{(2)})\sin\varphi)^2}}\right)^{\lambda}
\end{aligned}
\quad \text{(A-2)}
$$

and $\hat{N}_{12}$, $\hat{N}_{21}$, N_{kj} $(k,j = 1,2)$ are defined as

$$
\begin{aligned}
N_{11} &= \frac{\alpha_1+\beta_1\beta_2+\lambda_1\lambda_2}{1-\beta_1\beta_2-\lambda_1\lambda_2} \quad N_{22} = \frac{\alpha_2+\beta_1\beta_2+\lambda_1\lambda_2}{1-\beta_1\beta_2-\lambda_1\lambda_2} \\
N_{12} &= \frac{(1+\alpha_2)\beta_1\delta^{(2)^{-1}}}{1-\beta_1\beta_2-\lambda_1\lambda_2} \quad N_{21} = \frac{(1+\alpha_1)\beta_2\delta^{(2)}}{1-\beta_1\beta_2-\lambda_1\lambda_2} \\
\hat{N}_{12} &= \frac{(1+\alpha_2)\lambda_1\delta^{(2)^{-1}}}{1-\beta_1\beta_2-\lambda_1\lambda_2} \quad \hat{N}_{21} = \frac{(1+\alpha_1)\lambda_2\delta^{(2)}}{1-\beta_1\beta_2-\lambda_1\lambda_2}.
\end{aligned}
\quad \text{(A-3)}
$$

For results of $\mathbf{Q}(\lambda)$ for $\kappa^{(2)} < 1$ please refer to the thesis by Lin (1995).

APPENDIX B

The elements of matrix $\mathbf{Q}$ shown in Appendix A cannot be applied when material #2 is degenerated, i.e., $\kappa^{(2)} = 1$. Employing limiting process of L'Hospital rule, appropriate forms of the elements of matrix $\mathbf{Q}$ can be obtained as

$$
\begin{aligned}
\mathbf{Q}(1,1) =& (-\delta^{(2)^2}\sin^2\varphi-\cos^2\varphi)^{-1}\{-N_{11}[2\delta^{(2)}\lambda\sin\varphi\cos\varphi\sin(\lambda(\pi-2\varphi')) \\
& +\delta^{(2)^2}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(1-\lambda^2)+\cos(\lambda(\pi-2\varphi'))\cos^2\varphi]+N_{22}\delta^{(2)^2}\lambda^2\sin^2\varphi\cos(\lambda(\pi-2\varphi')) \\
& +N_{12}\delta^{(2)}\lambda[\sin(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi-\delta^{(2)}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(\lambda+1)] \\
& +N_{21}\delta^{(2)}\lambda[\sin(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi+\delta^{(2)}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(1-\lambda)] \\
& -\hat{N}_{12}\delta^{(2)}\lambda[-\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi-\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(\lambda+1)] \\
& +\hat{N}_{21}\delta^{(2)}\lambda[-\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi+\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1-\lambda)]\}
\end{aligned}
\quad \text{(B-1)}
$$

$$
\begin{aligned}
\mathbf{Q}(2,2) =& (-\delta^{(2)^2}\sin^2\varphi-\cos^2\varphi)^{-1}\{N_{11}\delta^{(2)^2}\lambda^2\sin^2\varphi\cos(\lambda(\pi-2\varphi')) \\
& -N_{22}[-2\delta^{(2)}\lambda\sin\varphi\cos\varphi\sin(\lambda(\pi-2\varphi')) \\
& +\delta^{(2)^2}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(1-\lambda^2)+\cos(\lambda(\pi-2\varphi'))\cos^2\varphi] \\
& +N_{12}\delta^{(2)}\lambda[-\sin(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi-\delta^{(2)}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(\lambda+1)] \\
& +N_{21}\delta^{(2)}\lambda[-\sin(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi+\delta^{(2)}\sin^2\varphi\cos(\lambda(\pi-2\varphi'))(1-\lambda)] \\
& -\hat{N}_{12}\delta^{(2)}\lambda[\cos(\lambda(\pi-2\varphi')\sin\varphi\cos\varphi-\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(\lambda+1)] \\
& +\hat{N}_{21}\delta^{(2)}\lambda[\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi+\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1-\lambda)]\}
\end{aligned}
\quad \text{(B-2)}
$$

$$
\begin{aligned}
\mathbf{Q}(1,2) = {} & (-\delta^{(2)^2}\sin^2\varphi - \cos^2\varphi)^{-1}\{-N_{11}\delta^{(2)^2}\lambda[-\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi \\
& -\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1+\lambda)] \\
& -N_{22}\delta^{(2)^2}\lambda[\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi-\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1+\lambda)] \\
& +N_{12}\delta^{(2)}\sin(\lambda(\pi-2\varphi'))[-\cos^2\varphi-\delta^{(2)^2}\sin^2\varphi(\lambda+1)]-N_{21}\delta^{(2)^3}\lambda^2\sin^2\varphi\sin(\lambda(\pi-2\varphi')) \\
& +\hat{N}_{12}\delta^{(2)}\cos(\lambda(\pi-2\varphi'))[-\cos^2\varphi-\delta^{(2)^2}\sin^2\varphi(\lambda+1)]\}+\hat{N}_{21}\delta^{(2)^3}\lambda^2\sin^2\varphi\cos(\lambda(\pi-2\varphi'))\}
\end{aligned} \tag{B-3}
$$

$$
\begin{aligned}
\mathbf{Q}(2,1) = {} & (-\delta^{(2)^2}\sin^2\varphi - \cos^2\varphi)^{-1}\{N_{11}\lambda[-\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi \\
& +\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1-\lambda)] \\
& +N_{22}\lambda[\cos(\lambda(\pi-2\varphi'))\sin\varphi\cos\varphi+\delta^{(2)}\sin^2\varphi\sin(\lambda(\pi-2\varphi'))(1-\lambda)] \\
& +N_{12}\delta^{(2)}\lambda^2\sin^2\varphi\sin(\lambda(\pi-2\varphi'))-N_{21}\delta^{(2)^{-1}}\sin(\lambda(\pi-2\varphi'))[-\cos^2\varphi-\delta^{(2)^2})\sin^2\varphi(\lambda-1)] \\
& +\hat{N}_{12}\delta^{(2)}\lambda^2\sin^2\varphi\cos(\lambda(\pi-2\varphi'))+\hat{N}_{21}\delta^{(2)^{-1}}\cos(\lambda(\pi-2\varphi'))[-\cos^2\varphi-\delta^{(2)^2}\sin^2\varphi(\lambda-1)]\}
\end{aligned} \tag{B-4}
$$

where

$$\varphi' = \tan^{-1}(\delta^{(2)}\tan\varphi) \quad 0 \leqslant \varphi' \leqslant \pi. \tag{B-5}$$